

Qinglin Roger Yu
Guizhen Liu

Graph Factors and Matching Extensions



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With 51 figures



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To Jenny, Barbara and Kyla

Preface

Graph theory is one of the branches of modern mathematics which has shown impressive advances in recent years. An explosive growth of graph theory is witnessed due to its essential roles providing structural models and indispensable tools in computer science, communication networks and combinatorial optimization problems.

Graph theory has matured mathematically as indicated by an increasing number of deeper results, such as the Regularity Lemma, Hall's Theorem, Graph Minor Theorem, which have been successfully applied to many branches of mathematics. In the meantime, graph theory has grown stronger by introducing techniques from other branches of mathematics (e.g., probabilistic method, linear algebra, group theory, and topology).

Matching theory, or more generally factor theory, is one of the fundamental areas in graph theory. It studies the structures and properties of matchings and factors — the simplest nontrivial substructures of graphs. Matchings and factors have many applications in other areas of graph theory, and techniques from factor theory such as the alternating path and decomposition procedure, are used in systematic approaches to combinatorial problems. Matching and factor theory is also one of earliest topics to be studied in graph theory.

Since the appearance of the classic work *Matching Theory* by Lovász and Plummer in 1986, factor theory has flourished over the last two decades, and much new and interesting progress has been made in graph factors and matching extension theory. However, these new results are not well summarized in traditional graph theory textbooks. This book is intended to serve as a collection of recent results in this interesting and very active field, which could serve as a reference manual for researchers or an introduction for young graph theorists or graduate students. To this end, sections on more advanced topics are included, and a number of interesting and challenging open problems and conjectures are presented at the end of each chapter.

For two reasons, we do not intend to cover all aspects of factor theory. One reason is that the branch has progressed so much that we are unable to include all the old and recent developments in it; the second reason is that

some topics are already covered elegantly in earlier books, e.g., Lovász and Plummer's book on matching theory. Our book concentrates mainly on the theoretical aspects of factor theory, not including the algorithmic aspects. Moreover, due to space limitation, some interesting and closely related topics were left out (e.g., factor factorizations, factors in random graphs, $(1, f)$ -odd-factor, subgraph packing problems, etc.), which can be found in the original literature. We give priority to the topics which are still very active but lack of exposures, such as component factors, connected factors, matching extension, optimal brick decompositions, fractional factors and L -factors.

This book is based on lecture notes written for summer graduate schools at Nankai University and Shandong University in 2005. The selection of the material was of course heavily influenced by our personal interests, as well as the limitations of space, while trying to cope with the recent development in the areas not covered by any known book. The book is primarily aimed at researchers and graduate students in graph theory. However, most of the material discussed is accessible to anyone with an undergraduate level understanding of mathematics. Our main source of materials is from three sources: research articles, textbooks and survey papers. The complete list of articles is given at the end of the book. The textbooks include:

1. J. Akiyama and M. Kano, *Factors and Factorizations of Graphs*, Version 1.0, June 2007 (in press).
2. J. A. Bondy and U. S. R. Murty, *Graph Theory*, Graduate Texts in Mathematics 244, Springer, 2008.
3. L. Lovász and M. D. Plummer, *Matching Theory*, North-Holland Inc., Amsterdam, 1986.

The survey papers include:

1. J. Akiyama and M. Kano, Factors and factorizations of graphs – a survey, *J. Graph Theory* 9 (1985) 1–42.
2. M. Kouider and P. D. Vestergaard, Connected factors in graphs – a survey, *Graphs Combin.* 21 (2005) 1–26.
3. M. D. Plummer, Extending matchings in graphs – a survey, *Discrete Math.* 127 (1994) 277–292.
4. M. D. Plummer, Extending matchings in graphs – an update, *Congressus Numerantium* 116 (1996) 3–32.
5. M. D. Plummer, Graph factors and factorization, *Handbook on Graph Theory*, Eds.: J. Gross and R. Yellen, CRC Press, New York, 2003 403–430.

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We welcome any feedback regarding this book, corrections, and comments. Feel free to reach us by e-mail at yu@tru.ca (Yu) or gzliu@sdu.edu.cn (Liu). We will maintain a web site to provide an update list of errata and corrections. The Web page is

<http://www.tru.ca/faculty/yu/factor.book.html>

Thanks also to Mr. Tianfu Zhao, our editor at Higher Education Press for his expertise, dedication and encouragement. Our gratitude also goes to the editors at Springer for their professional assistants.

And finally, thanks to our family, who give all support and patient while we carried out this project.

Jinan,
May 2009

Qinglin Roger Yu
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Chapter 1

Matchings and Perfect Matchings

As a young branch of mathematics, Graph Theory has experienced the explosion growth as the same phenomenon that has been taking place in computing science and communication networking. In the mean time, there are many new terminologies and knowledge accumulated in the process. So there are often more than one names or notions defined for a same entity. We list the terms and notions frequently used in this book and hope to provide readers with a consistent reference.

Notions

Let G be a graph, with vertex set $V(G)$ and edge set $E(G)$. Let f be a positive integer-valued function defined on vertex set $V(G)$.

$|S|$: order of a set S ;

$\lfloor x \rfloor$: the largest integer not greater than x ;

$\lceil x \rceil$: the smallest integer not less than x ;

$V(G)$: the vertex set of G , sometimes we use $|G|$ for the order of G , i.e., $|G| = |V(G)|$;

$E(G)$: the edge set of G , occasionally we write $\|G\|$ for the size of G , i.e., $\|G\| = |E(G)|$;

$A \triangle B$: symmetric difference of two sets A and B , i.e., $A \triangle B = (A - B) \cup (B - A)$;

$d_G(v)$: the degree of a vertex v in G ;

$\delta(G)$: the minimum degree of G ;

$\Delta(G)$: the maximum degree of G ;

$\nabla(v)$: the edges incident with the vertex v ;

$\nabla(S)$: the set of edges with exactly one end-vertex in the set S ;

$N_G(S)$: the set of vertices adjacent to a vertex of S , i.e., $N_G(S) = \{x \mid xy \in E(G) \text{ and } y \in S\}$, or in short $N(S)$ if no confusion is arisen;

$E_G(S, T)$: the edges with one end in S and another end in T and $e_G(S, T) = |E_G(S, T)|$;

- $\sigma_k(G)$: degree sum of k independent vertices;
- $\mu(G)$: the matching number of G ;
- $\mu_f(G)$: the fractional matching number of G ;
- $\eta(G)$: the maximum cardinality of joins in G ;
- $\xi(\Sigma)$: the smallest integer m such that no graph G embeddable in surface Σ is m -extendable;
- $ext(G)$: the extendability number of a graph G ;
- $b(G)$: the number of bricks of G in a decomposition;
- $\hat{c}(G)$: the number of critical $\{K_2, K_3\}$ -components in G ;
- $p(G)$: the number of Petersen subgraphs;
- $i\mu(G)$: the induced matching number;
- G^c : the complement of G ;
- $G[S]$: the induced subgraph of G by a vertex set S ;
- $o(G)$: the number of odd components of G ;
- $\omega(G)$: the number of components of G ;
- $c(G)$: the circumference of G ;
- $\beta^*(G)$: the vertex covering number of G ;
- $I(G)$: the set of isolated vertices in G ;
- $i(G)$: the number of isolated vertices in G ;
- $D(G)$: the set of all vertices in G which are not saturated by at least one maximum matching of G ;
- $A(G)$: the set of vertices in $V(G) - D(G)$ adjacent to at least one vertex in $D(G)$;
- $C(G) = V(G) - D(G) - A(G)$;
- $fc(G)$: the number of factor-critical components in G ;
- $tc(G)$: the number of triangle clusters in G ;
- $oc(G)$: the number of those components of G that are odd cacti;
- $\Phi(G)$: the number of 1-factors in G ;
- G_δ : the subgraph induced by the vertices with minimum degree in G ;
- G_Δ : the subgraph induced by the vertices with maximum degree in G ;
- $\kappa(G)$: the vertex connectivity of G ;
- $\kappa'(G)$: the edge connectivity of G ;
- $\lambda(C)$: the characteristic of a cut C ;
- $c\lambda(G)$: the cyclic connectivity of G ;
- $\alpha(G)$: the independent number of G ;
- $\gamma(G)$: the dominating number;
- $\gamma^*(G)$: the genus of G ;
- $\chi'(G)$: the edge chromatic number of G ;
- $\chi'_c(G)$: the edge-cover number of G ;
- $\tau_2(G)$: the size of a minimum 2-covering of G ;
- $G \vee H$: the join of two graphs G and H ;
- G^r : the r th power of a graph G ;
- $t(G)$: the toughness of G introduced by Chvátal [151],

$$t(G) = \min \left\{ \frac{|S|}{\omega(G-S)} \mid S \subseteq V(G), \omega(G-S) \geq 2 \right\};$$

$\tau(G)$: a variation of toughness introduced by Enomoto [184],

$$\tau(G) = \min \left\{ \frac{|S|}{\omega(G-S) - 1} \mid S \subseteq V(G), \omega(G-S) \geq 2 \right\}.$$

We define, for all disjoint subsets S and T of $V(G)$,

$$\delta_G(S, T) =$$

$$\begin{cases} k|S| + \sum_{x \in T} d_G(x) - k|T| - e_G(S, T) - q_G(S, T) & \text{for } k\text{-factor;} \\ f(S) + \sum_{x \in T} d_G(x) - f(T) - e_G(S, T) - q_G(S, T) & \text{for } f\text{-factor;} \\ f(S) + \sum_{x \in T} d_G(x) - g(T) - e_G(S, T) - q_G(S, T) & \text{for } (g, f)\text{-factor,} \end{cases}$$

where $q_G(S, T)$ is defined for k -factors, f -factors and (g, f) -factors in Theorems 2.1.1, 2.1.2 and 2.1.3, respectively.

1.1 Definitions and terminologies

To understand the developments of matchings and factors in graphs, we would like to review the history briefly by listing the most significant milestones in this fascinating area of graph theory.

The first attempt on the study of factors was made by Danish mathematician Petersen (1891), who proved that every graph of even degrees can be decomposed into the union of edge-disjoint 2-factors. This was motivated from the study of an algebraic factorization problem. He also showed that every 2-connected cubic graph has a 1-factor. These two results can be viewed as a forerunner of modern graph factor theory.

For matchings in bipartite graphs, König (1931) and Hall (1935) obtained the so-called König-Hall Theorem (sometimes, known as Hall's Theorem). Due to its wide applications in many graph theory problems and other branches of mathematics, König-Hall Theorem remains one of most influential graph-theoretic results.

In 1947, Tutte gave a characterization (i.e., so-called Tutte's 1-Factor Theorem) for the existence of perfect matchings in arbitrary graphs and it has become a cornerstone of factor theory. Till now, this elegant theorem is still one of the most fundamental results in factor theory. Subsequently, Tutte (1952) extended the techniques in the proof of 1-Factor Theorem to obtain a sufficient and necessary condition for a graph to have an f -factor.

Gallai (1964) and Edmonds (1965), independently, investigated a canonical decomposition of arbitrary graphs in terms of its maximum matchings

and thus revealed the structure of graphs related to matchings. This deep and important result is referred as Gallai-Edmonds Structure Theorem. The existence of the canonical decomposition and an efficient method to find such a decomposition allowed Edmonds to obtain the first polynomial algorithm for finding maximum matchings in graphs.

The most general degree-constrained factors, (g, f) -factors, were studied by Lovász (1970). He gave necessary and sufficient conditions for a graph to have a (g, f) -factor. This theorem generalized the criteria of all other factors, such as 1-factors, k -factors, f -factors and $[a, b]$ -factors.

For the more comprehensive account of history on matching theory and graph factors, readers can refer to Preface of Lovász and Plummer's *Matching Theory* or Biggs, Lloyd and Wilson's *Graph Theory 1736-1936*.

In this book, we mainly deal with factors in *finite undirected simple* graphs. Some results also hold for graphs with multiple edges, which we point out accordingly.

The *degree* of a vertex v in a graph G , denoted by $d_G(v)$, is the number of edges of G incident with v , each loop counting as two edges. In particular, if G is a simple graph, $d_G(v)$ is the number of neighbors of v in G . A vertex of degree zero or one are called an *isolated vertex* and a *leaf*, respectively. A *tree* is a connected graph containing no cycles. Thus every tree has at least two leaves. It is well-known that every connected graph G contains a tree. For a tree T in a connected graph, we always assume that T is a *spanning* connected subgraph without cycle.

We denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degrees of the vertices of G .

A *cut-edge* of a graph is one whose deletion results in one more component. Sometimes, a cut-edge is also called a *bridge*. Clearly, every edge of a tree is a cut-edge.

A *matching* M of a graph G is a subset of $E(G)$ such that any two edges of M have no end-vertices in common. A matching of k edges is called a *k-matching*. Let d be a non-negative integer. A matching is called a *defect- d matching* if it covers exactly $|V(G)| - d$ vertices of G . A defect-0 matching is called a *perfect matching* and defect-1 matching is called *near-perfect matching*.

A perfect matching is also referred as *1-factor* since it is a 1-regular subgraph. If a graph has 1-factors, it is called *1-factorable*. There are probably equal numbers of people using “perfect matchings” rather than “1-factors”, so we will use them non-discriminatively.

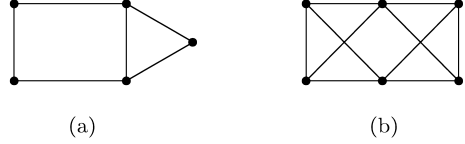
If an edge e is contained in a 1-factor, we called e an *allowed edge*. If an edge f does not lie in any 1-factor, then f is *forbidden*.

A *covering* of a graph G is a subset S of $V(G)$ such that every edge of G has at least one end in S . A covering S^* is a *minimum covering* if G has

no covering S with $|S| < |S^*|$. The order of a minimum covering is called *covering number* and denoted by $\beta^*(G)$.

A graph G is said to be *bicritical* if for every pair of distinct vertices u and v , $G - \{u, v\}$ has a 1-factor. A graph G is *factor-critical* if $G - u$ has a 1-factor for every $u \in V(G)$ (see Fig. 1.1).

Fig. 1.1 (a) a factor-critical graph and (b) a bicritical graph.



A graph G is called *n -factor-critical* if the subgraph $G - S$ has a 1-factor for all n -subset S of $V(G)$.

Clearly, the concept of n -factor-criticality is a generalization of bicriticality and factor-criticality. That is, factor-critical graphs and bicritical graphs are n -factor-critical graph when $n = 1$ and 2, respectively.

From the definitions, we can see the following result easily.

Proposition 1.1.1.

- (a) A graph G is factor-critical if and only if the join of G and K_1 , $G \vee K_1$, is bicritical;
- (b) a graph G is bicritical if and only if $G - v$ is factor-critical for every $v \in V(G)$.

Let G be a connected graph with a 1-factor and $|V(G)| \geq 2k + 2$. If each k -matching of G is contained in a 1-factor, we call G a *k -extendable* graph. For convenience, a 0-extendable graph means a graph which has a 1-factor.

Clearly bicritical graphs are 1-extendable. A 3-connected bicritical graph is called a *brick*. A *brace* is a 2-extendable bipartite graph (i.e., 3-connected 2-extendable bipartite graph). The examples of a brick and a brace are given in Fig. 1.2.

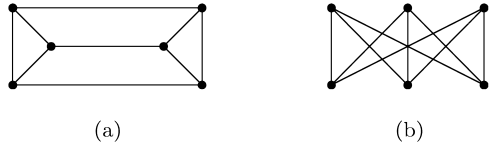


Fig. 1.2 (a) a brick and (b) a brace.

Factors in a graph are spanning subgraphs satisfying some given properties. A *k -factor* of a graph G is a spanning subgraph H such that $d_H(x) = k$ for every $x \in V(G)$.

Let f be an integer-valued function defined on $V(G)$, i.e.,

$$f : V(G) \rightarrow \mathbb{Z}^+ = \{0, 1, 2, \dots\},$$

then a spanning subgraph H such that $d_H(x) = f(x)$ for all $x \in V(G)$ is called an f -factor.

A more general factor is so-called (g, f) -factors: for two functions $g, f : V(G) \rightarrow \mathbb{Z}$, a spanning subgraph H is called a (g, f) -factor if $g(x) \leq d_H(x) \leq f(x)$ for all $x \in V(H)$ (see an example in Fig. 1.3). In addition, if $g(x) \equiv f(x) \pmod{2}$ for all $x \in V(G)$, a (g, f) -factor F with $d_F(x) \equiv f(x) \pmod{2}$ is called a (g, f) -parity-factor.

For two positive integers a, b , let $g(x) = a$ and $f(x) = b$ for all vertices $x \in V$. Then (g, f) -factors are called $[a, b]$ -factors.

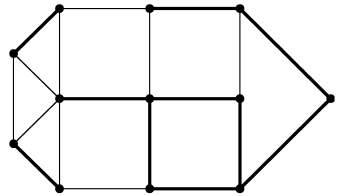


Fig. 1.3 A (g, f) -factor with $g(x) = 1$ and $f(x) = 3$ for all $x \in V$.

A set $S \subset V$ is a *dominating set* of G if every vertex in V is either in S or is adjacent to a vertex in S . If A and B are subsets of V , we say that A *dominates* B if every vertex of B has a neighbor in A or is a vertex of A ; we also say that B is *dominated* by A . The *dominating number*, $\gamma(G)$, of G is the minimum cardinality of a dominating set. A set S is *2-dominating* if every vertex of V is either in S or is within distance 2 to S .

We only define the most often used terms of the book in this section. For other undefined terms, we follow Bollobás [83].

1.2 Matchings in bipartite graphs and augmenting path

In this section, we focus on matchings and perfect matchings in bipartite graphs, which is the starting point of matching theory. This is not surprising since matchings in bipartite graphs are closely related to other disciplines of mathematics (e.g., matrix theory and set theory) and many application problems (e.g., the job assignment problem). Furthermore, the properties and techniques of matchings in bipartite graphs also motivate the research of matchings in non-bipartite graphs.

For a matching M , a vertex x of G is called *saturated* or *covered* by M if x is incident to an edge of M . A matching M is called a *maximum matching* if $|M| \geq |M'|$ for any matching M' of G . Clearly, a perfect matching (i.e.,

a 1-factor) is a maximum matching saturating every vertex of a graph. Of course, any graph with perfect matchings must have even number of vertices.

In many applications, one wishes to find a matching in a bipartite graph $G = (X, Y)$ which covers every vertex in X . Necessary and sufficient conditions for the existence of such a matching were proven by Hall (1935). To this day, Hall's Theorem remains one of the most widely applied graph-theoretic results. The proof here is due to Rado.

Theorem 1.2.1 (Hall's Theorem, Hall (1935)). *Let $G = (X, Y)$ be a bipartite graph. Then G has a matching saturating all vertices of X if and only if for any $S \subseteq X$,*

$$|N(S)| \geq |S|. \quad (1.1)$$

Proof. The necessity is clear.

Let G be a minimal bipartite graph satisfying the condition (1.1). It suffices to show that G consists of $|X|$ independent edges.

Otherwise, G contains two edges, x_1y and x_2y , where $x_1, x_2 \in X$, $x_1 \neq x_2$ and $y \in Y$. From the minimality, the deletion of either of these edges invalidates (1.1). Thus there are two subsets $X_1, X_2 \subset X$ such that for $i = 1, 2$ we have $|N(X_i)| = |X_i|$, and x_i is the only vertex of X_i adjacent to y . Then

$$\begin{aligned} |N(X_1) \cap N(X_2)| &\geq |N(X_1 - \{x_1\}) \cap N(X_2 - \{x_2\})| + 1 \\ &\geq |N(X_1 \cap X_2)| + 1 \geq |X_1 \cap X_2| + 1. \end{aligned}$$

But this implies the following contradiction:

$$\begin{aligned} |N(X_1 \cup X_2)| &= |N(X_1) \cup N(X_2)| \\ &= |N(X_1)| + |N(X_2)| - |N(X_1) \cap N(X_2)| \\ &\leq |X_1| + |X_2| - |X_1 \cap X_2| - 1 \\ &= |X_1 \cup X_2| - 1. \end{aligned}$$

□

Note that Hall's Theorem also holds for multigraphs. The above result is the usual version of matching characterization in bipartite graphs. However, we present the next version in terms of isolated vertices, which has a strong similarity to Tutte's 1-Factor Theorem referred to later. Recall that $i(G)$ denotes the number of isolated vertices in G .

Theorem 1.2.2 (Hall's Matching Theorem, Hall (1935)). *Let $G = (X, Y)$ be a bipartite graph. Then G has a perfect matching if and only if $|X| = |Y|$ and for any $S \subseteq X$,*

$$i(G - S) \leq |S|.$$

Remark 1.2.1. For every bipartite graph $G = (X, Y)$, with $|X| \geq |Y|$ and maximum degree Δ , König (1916) showed that there exists Δ -regular bipartite supergraph $G' = (X, Y')$ (a multigraph is allowed) such that $Y \subseteq Y'$ and G is a subgraph of G' .

Corollary 1.2.1 (König (1916)). *Every r -regular bipartite graph G can be decomposed into r edge-disjoint 1-factors. In particular, G has a 1-factor.*

Corollary 1.2.2. *Every bipartite graph G has a maximum matching which covers all the vertices of maximum degree.*

Proof. By Remark 1.2.1, there exists a $\Delta(G)$ -regular supergraph G' containing G . From Corollary 1.2.1, G' has a 1-factor and so G has a matching covers all the vertices of maximum degree. In turn, this matching can be augmented to a maximum matching of G . $\square$

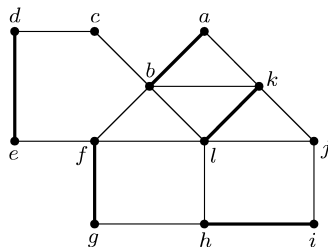
A family of graphs which share many properties with bipartite graphs was discovered by Fulkerson, Hoffman and McAndrew [219]. A graph G is said to have the *odd cycle property* (OCP in short) if every pair of odd cycles in G either has a vertex in common or is joined by an edge. We shall discuss the relation between this property and the existence of factors in much more details in Section 2.2.

The following is a generalization of Corollary 1.2.1.

Theorem 1.2.3. *If G is a r -regular graph of even order with the odd cycle property, then G has a 1-factor.*

Let M be a matching in a graph G . An M -alternating path or cycle in G is a path or cycle whose edges are alternately in M and $E(G) - M$. Note that an M -alternating path might or might not start or end with edges of M . If neither its origin nor its terminus is covered by M , the M -alternating path is called an M -augmenting path (see Fig. 1.4).

Fig. 1.4 An M -alternating cycle $C = ablk$, an M -alternating path $P = defghi$ and an M -augmenting path $P' = cdefghij$.



The idea of M -augmenting path is first introduced by Berge (1957). Today, it has become a very matured and standard technique in the study of factor theory. There are many variations of augmenting path defined for different

problems, we shall see all kinds of variations in the coming chapters. It is not an overstatement that augmenting path technique is the most powerful technique not only in the study of matchings but also for the study of all factors.

Berge (1957) used augmenting path to give a characterization of maximum matching.

Theorem 1.2.4 (Berge (1957)). *A matching M in a graph G is maximum if and only if G has no M -augmenting paths.*

Proof. Let M be a matching in G . Suppose that G contains an M -augmenting path P . Then $M' := M \triangle E(P)$ is a matching in G , and $|M'| = |M| + 1$. Thus M is not a maximum matching.

Conversely, suppose that M is not a maximum matching, and let M'' be a maximum matching in G , then $|M''| > |M|$. Set $H := G[M \triangle M'']$. Then each vertex of H has degree one or two in H , for it can be incident with at most one edge of M and one edge of M'' . Consequently, each component of H is either an even cycle with edges alternately in M and M'' , or else a path with edges alternately in M and M'' .

Because $|M''| > |M|$, the subgraph H contains more edges of M'' than of M , and therefore some path-component P of H must start and end with edges of M'' . The origin and terminus of P , being covered by M'' , are not covered by M . The path P is thus an M -augmenting path in G , a contradiction. $\square$

Hall's Theorem can be restated in a more general version. We denote the matching number, the size of a maximum matching, by $\mu(G)$.

Theorem 1.2.5. *Let $G = (X, Y)$ be a bipartite graph, M a matching in G , and U the set of M -unsaturated vertices in X . Then*

- (a) *for any subset S of X , $|U| \geq |N(S)| - |S|$; and*
- (b) *$|U| = |N(S)| - |S|$ if and only if M is a maximum matching of G .*

Furthermore, the matching number of G is given by

$$\mu(G) = |X| - \max\{|S| - |N(S)| \mid S \subseteq X\}.$$

The expression for $\mu(G)$ is known as the König-Ore Formula.

Next, we investigate the properties of matching transformation in bipartite graphs. That is, given two matchings, we can start with one and transform gradually to another one through the operation of symmetric difference.

Theorem 1.2.6. *A maximum matching M of a graph G can be obtained from any other maximum matching M' by a sequence of transfers along alternating cycles and paths of even lengths.*

Proof. From the proof of Theorem 1.2.4, every component of $G[M' \triangle M]$ is either an alternating cycle or an alternating even path relative to M' . Changing M' in each component in turn will transform M' into M . $\square$

Theorem 1.2.7 (Dulmage and Mendelsohn [435]). *Let $G = (X, Y)$ be a bipartite graph and M_1, M_2 be two matchings in G . Then there is a matching $M \subseteq M_1 \cup M_2$ such that M covers all the vertices of X saturated by M_1 and all the vertices of Y saturated by M_2 , that is, $V(M) \supseteq (V(M_1) \cap X) \cup (V(M_2) \cap Y)$.*

Proof. Let X_i (respectively, Y_i) be the vertices of X (of Y) covered by M_i ($i = 1, 2$). Let $G_1, G_2, \dots, G_k$ be the connected components of $G[M_1 \Delta M_2]$. Then each G_i is an even cycle or path. Let $M_{1i} = E(G_i) \cap M_1$ and $M_{2i} = E(G_i) \cap M_2$ ($1 \leq i \leq k$) and define a matching P_i in each G_i :

$$P_i = \begin{cases} M_{1i} & \text{if } G_i \text{ is a cycle;} \\ M_{1i} & \text{if there is a vertex } v \in V(G_i) \cap (X_1 - X_2); \\ M_{2i} & \text{if there is a vertex } v \in V(G_i) \cap (Y_2 - Y_1). \end{cases}$$

It is not hard to check that $M = (M_1 \cap M_2) \cup P_1 \cup P_2 \cup \dots \cup P_k$ is the required matching. $\square$

The local neighborhood condition can be used as a sufficient condition for the maximum matching in bipartite graphs. The following theorem is due to Alon.

Theorem 1.2.8. *Let $G = (X, Y)$ be a bipartite graph. If one of the following conditions holds,*

- (a) *$d(x)$ is odd for all $x \in X$ and any two vertices of X have an even number of common neighbors; or*
- (b) *$d(x) \geq 1$ for all $x \in X$ and $d(x) \geq d(y)$ for all $xy \in E(G)$, where $x \in X$ and $y \in Y$,*

then G has a matching saturating X .

The next result is of type of degree condition.

Theorem 1.2.9. *Let $G = (X, Y)$ be a bipartite graph with $|X| = |Y| = n$. If $\delta(G) \geq \frac{n}{2}$, then G has a 1-factor.*

A matching is *induced* if no two edges in the matching are joined by an edge in G . Unlike the usual matching problem, the induced matching is a much more difficult problem to deal with. The problem of finding a maximum induced matching is NP-hard, even for bipartite graphs and for planar graphs of maximum degree 4 (see [110]). The polynomial algorithms for solving the maximum induced matching problem are found only for a few families of special graphs (e.g., weakly chordal graphs, circle graphs, chordal graphs, and outerplanar graphs). However, we have a neat formula for the size of maximum induced matching, i.e., induced matching number $i\mu(G)$, in bipartite graphs.

Theorem 1.2.10 (Liu and Zhou [376]). *Let $G = (X, Y)$ be a connected bipartite graph. Then the size of a maximum induced matching M of G is*

$$i\mu(G) = \max\{|S| \mid S \subseteq X \text{ such that } N_G(T) \neq N_G(S) \text{ for all } T \subset S\}.$$

Matchings in bipartite graphs and Hall's Theorem have proved to be valuable tools both in graph theory and in other areas of mathematics. In fact, Hall's Theorem has several equivalent formulations, including the following one in terms of set systems.

Let $\mathcal{A} = (A_i \mid i \in I)$ be a finite family of (not necessarily distinct) subsets of a finite set A , where $i \in I$ is any index set. A *system of distinct representatives* (SDR) for the family $\mathcal{A}$ is a set $\{a_i \mid i \in I\}$ of distinct elements of A such that $a_i \in A_i$ for all $i \in I$. In this language, Hall's Theorem says that $\mathcal{A}$ has a system of distinct representatives if and only if $|\cup_{i \in J} A_i| \geq |J|$ for all subsets J of I . To see that this is indeed a reformulation of Hall's Theorem, let $G = (X, Y)$, where $X = I$, $Y = A$, and $E(G) = \{ia \mid a \in A_i, \text{ where } i \in I \text{ and } a \in A\}$.

The other formulations of Hall's Theorem can be found in $(0, 1)$ -matrix theory and transversal matroid.

1.3 Matchings in non-bipartite graphs

In this section, we consider the existence of 1-factors in arbitrary graphs. The characterization of 1-factors was given by Tutte in 1947. It is one of the most important theorems in graph theory for several reasons: firstly 1-factor as a 1-regular spanning subgraph is a simplest but the most essential structure of any graph; secondly it has wide applications in other areas of mathematics and in optimization problems (it often serves as subroutine in many optimization algorithms); finally it is one of earliest topics in graph theory which has a relative complete understanding and matured techniques compared to other topics in graph theory.

A component C of a graph is called *odd* if $|V(C)|$ is odd; otherwise, it is called *even*. Recall that $o(H)$ is the number of odd components in a graph H . Using a parity argument, it is not hard to see that

$$o(G - S) + |S| \equiv |V(G)| \pmod{2}. \quad (1.2)$$

If G is of even order, then $|V(G)| \equiv 0 \pmod{2}$ and thus

$$o(G - S) \equiv |S| \pmod{2}. \quad (1.3)$$

This fact is quite handy when estimating the number of odd components in a graph.

Now we present Tutte's criterion for 1-factors. Besides Tutte's original proof for this theorem, there are several shorter and simpler proofs, respectively, from Anderson (1971), Lovász (1975), as well as Ore's transformation from bipartite graphs. Here we use the proof from Lovász.

Theorem 1.3.1 (Tutte's 1-Factor Theorem, Tutte (1947)). *A graph G has a 1-factor if and only if for any $S \subseteq V(G)$*

$$o(G - S) \leq |S|. \quad (1.4)$$

Proof. Necessity. Since the odd components of $G - S$ must have vertices matched to distinct vertices of S , we have $o(G - S) \leq |S|$.

Sufficiency. Note that adding an edge joining two components of $G - S$ does not increase the number of odd components. Hence (1.4) is preserved from the addition of edges. That is, let $G' = G \cup e$ and $S \subseteq V(G)$, then $o(G' - S) \leq o(G - S) \leq |S|$. Furthermore, if G' has no 1-factors, then G has no 1-factors either.

Assume that G satisfies (1.4) but has no 1-factors. Without loss of generality, let G be (edge-wise) maximal such a graph. Note that (1.4) implies that $|V(G)|$ is even (considering $S = \emptyset$).

Let S_0 be the set of vertices in G that have degree $|V(G)| - 1$.

Case 1. $G - S_0$ consists of disjoint complete graphs.

In this case, vertices in each component of $G - S_0$ can be paired in any way, with one extra in the odd components. Since $o(G - S_0) \leq |S_0|$ and each vertex of S_0 is adjacent to all of $G - S_0$, we can match the leftover vertices to vertices of S_0 . So unmatched vertices belong to S_0 , which induces a clique. Since $|V(G)|$ is even, by parity, the number of unmatched vertices in S_0 is even and thus we can form a 1-factor.

Case 2. $G - S_0$ is not a disjoint union of cliques.

In this case, $G - S_0$ has two vertices at distance 2; these are nonadjacent vertices x, z with a common neighbor $y \notin S_0$. Furthermore, $G - S_0$ has another vertex w not adjacent to y , since $y \notin S_0$. By the maximality of G , adding an edge to G creates a 1-factor; let M_1 and M_2 be 1-factors in $G \cup xz$ and $G \cup yw$, respectively. It suffices to show that $M_1 \triangle M_2$ contains a 1-factor avoiding xz and yw , because this will be a 1-factor of G .

Let $F = M_1 \triangle M_2$. Since $xz \in M_1 - M_2$ and $yw \in M_2 - M_1$, both xz and yw are in F . Since every vertex of G has degree 1 in each M_1 and M_2 , every vertex of G has degree 0 or 2 in F . Hence the components of F are even cycles and isolated vertices.

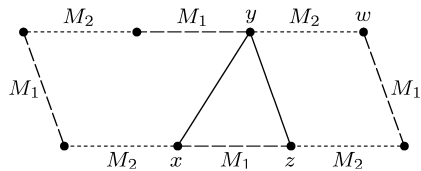


Fig. 1.5 Alternative paths.

Let C be the cycle of F containing xz (see Fig. 1.5). If C does not also contain yw , then the desired 1-factor consists of the edges of M_2 from C and all of M_1 not in C . If C contains both yw and xz , then to avoid them we use yx or yz . In the portion of C starting from y along yw , we use edges of M_1 to avoid using yw . When we reach $\{x, z\}$, we use zy if we arrive at z ; otherwise, we use xy . In the remainder of C we use the edges of M_2 . We have produced a 1-factor M' of C that does not use xz or yw . Combining M' with the edges of M_1 or M_2 outside C forms a 1-factor of G . $\square$

Hall's Theorem and Tutte's 1-Factor Theorem are in fact equivalent. They both belong to a large family of fundamental results in graph theory which are all equivalent to each other (see Jacobs and Heiratssatz [261]).

As one of applications of 1-Factor Theorem and also a refinement, Theorem 1.3.1 can also be restated in terms of factor-critical graphs. This result was a by-product of the algorithm for finding a maximum cardinality matching by Edmonds [165]. A non-algorithmic proof of this theorem was given by Anderson [54].

Let $fc(G)$ be the number of factor-critical components of graph G .

Theorem 1.3.2 (Edmonds [165]). *A graph G of even order has a 1-factor if and only if for every subset $S \subseteq V(G)$,*

$$fc(G - S) \leq |S|.$$

Proof. Since every factor-critical component is an odd component, so the necessity follows immediately from 1-Factor Theorem.

Conversely, assume that G has no 1-factors. Then, by 1-Factor Theorem, there exists a subset $S \subseteq V(G)$ such that $o(G - S) > |S|$ and $S \neq \emptyset$ or $V(G)$. Note the fact

$$o(G - S') \leq |S'| \quad \text{for all } S_0 \subset S' \subseteq V(G). \quad (1.5)$$

Choose *maximal* such a set S_0 . Then the maximality of S_0 implies that $G - S_0$ has no even components. Next we show that every component of $G - S_0$ is factor-critical, i.e., $fc(G - S_0) = o(G - S_0)$.

Let C be an odd component of $G - S_0$, and v any vertex of C . Then for every $T \subseteq V(C - v)$, (1.5) implies

$$\begin{aligned}
|S_0| + 1 + |T| &\geq o(G - (S_0 \cup \{v\} \cup T)) \\
&= o(G - S_0) - 1 + o(C - (\{v\} \cup T)) \\
&> |S_0| - 1 + o((C - v) - T).
\end{aligned}$$

Hence $o((C - v) - T) < |T| + 2$, which implies $o((C - v) - T) \leq |T|$ by (1.4). Therefore $C - v$ has a 1-factor by 1-Factor Theorem, or C is factor-critical. Consequently, $fc(G - S_0) = o(G - S_0) > |S_0|$, a contradiction to the assumption of the theorem. $\square$

In fact, Tutte's 1-Factor Theorem can be refined further by considering only the special subsets S (i.e., barrier set) and the number of factor-critical components. More refinement, i.e., Gallai-Edmonds Structure Theorem, will be discussed in Section 1.5.

When applying 1-Factor Theorem to various classes of graphs, the criterion can be much simpler and nicer. The next result is such an example and it can be proven easily by induction.

Proposition 1.3.1 (Chungphaisan (1976)). *A tree T of even order has a 1-factor if and only if $o(T - v) = 1$ for every vertex v of T .*

Augmenting path (or chain) technique lies in the heart of factor theory. It is an effective way to adjust a subgraph to get a better subgraph (or factor). Its significance is also reflected in the fact that if an augmenting technique is developed for a given factor problem (e.g., 1-factors, k -factors, f -factors, L -factors with $(*)$ property (refer to Section 2.4), etc.), then it often can be converted into a polynomial algorithm for the problem.

Augmenting path (or chain) technique poses many nice properties, as shown in the next theorem. These properties are important for the adjustment process to reach the optimal subgraphs. The uttermost usage of such properties can be found in treating L -factors in [401; 402].

For a matching M and a vertex set S , if $V(M) \cap S = \emptyset$, then we say that M *avoids* S .

Theorem 1.3.3 (see [14]). *Let A and B be vertex subsets of a graph G such that $|A| < |B|$. Then*

- (a) *if there exist two matchings, one saturating A and the other saturating B , then there exists a matching that saturates A and at least one vertex in $B - A$;*
- (b) *if there exist two maximum matchings, one avoiding A and the other avoiding B , then there exists a maximum matching that avoids A and at least one vertex in $B - A$.*

Proof. (a) Let M_A and M_B be matchings in G which saturate A and B , respectively. If M_A saturates one vertex in $B - A$, then M_A itself is the desired matching. Hence we may assume that M_A avoids $B - A$.

Consider the subgraph $G[M_A \triangle M_B]$. Then for every vertex b in $B - A$, there exists a path in $G[M_A \triangle M_B]$ starting with b , which may end with a vertex in $A - B$. Since $|A| < |B|$ and no path of $G[M_A \triangle M_B]$ ends with a vertex in $A \cap B$, there exists a path P in $G[M_A \triangle M_B]$ that starts with $b_1 \in B - A$ and ends with $x \notin A$. Since P is an M_A -alternating path connecting b_1 and x , $M_A \triangle E(P)$ is the desired matching in G , which saturates A and $b_1 \in B - A$.

(b) Let N_A and N_B be maximum matchings in G which avoid A and B , respectively. We may assume that N_A saturates $B - A$ since otherwise N_A itself is the desired matching. Then by the same argument as in (a), there exists a N_A -alternating path P in $G[N_A \triangle N_B]$ starting with a vertex $b \in B - A$ and ending with a vertex not contained in A . If both end-edges of P are contained in N_A , then $N_B \triangle E(P)$ is a matching with $||N_B|| + 1$ edges, contrary to the maximality of N_B . Thus one end-edge of P belongs to N_B , and thus $N_A \triangle E(P)$ is the desired maximum matching in G , which avoids A and $b \in B - A$. $\square$

To find a matching containing a given set of vertices, we have the following interesting result.

Theorem 1.3.4 (Kierstead [315]). *Let G be a graph and T the set of vertices of degree $\Delta(G)$. If $G[T]$ is a bipartite subgraph, then there exists a matching in G saturating all vertices of T .*

To conclude this section, we present a conjecture due to Macajova and Skoviera.

Conjecture 1.3.1 (Macajova and Skoviera (2005)). *Every 2-edge-connected cubic graph has two perfect matchings M_1 and M_2 such that $M_1 \cap M_2$ does not contain an odd edge-cut.*

1.4 Sufficient conditions for 1-factors

Although Tutte's 1-Factor Theorem provides a nice and simple criterion for 1-factors, to check the condition (1.4) is a non-trivial task. So many researchers have attempted to find simple sufficient conditions for the existence of 1-factors using various graphic parameters, or to determine special classes of graphs to be 1-factorable. The graphic parameters that have been studied include connectivity, claw-free, neighborhood, degree-sequence, toughness, binding number, etc. Those approaches not only provide a simple conditions for existence of 1-factors, but also enrich our understanding to the structures of graphs with 1-factors; more importantly it supplies many fresh ideas for

the study of other factors. In fact, many of them have been generalized to k -factors, $[a, b]$ -factors and so on. We will examine it in more details in later chapters, for instance, toughness in Sections 3.1 and 3.2, binding number in Section 3.3 and connectivity in Section 3.4.

When we use 1-Factor Theorem to show the existence of 1-factors or maximum matchings in a graph, often we have to deal with the number of odd components and count the number of edges between odd components and a vertex set. Besides the relation (1.2), the following result will be handy for such estimations, in particular, in regular graphs.

Proposition 1.4.1. *Let G be a r -regular graph ($r \geq 2$), and $S \subseteq V(G)$. Then for each odd component C of $G - S$,*

$$e_G(C, S) \equiv r \pmod{2}.$$

In particular, if G is an $(r-1)$ -edge-connected r -regular graph, then $e_G(C, S) \geq r$.

Proof. Since $|C|$ is odd, we have

$$r \equiv r|C| = \sum_{u \in V(C)} d_G(u) = e_G(C, S) + 2|E(C)| \equiv e_G(C, S) \pmod{2}.$$

Moreover, if G is an $(r-1)$ -edge-connected r -regular graph, then $e_G(C, S) \geq r-1$, and by combining this inequality and parity, we have $e_G(C, S) \geq r$. $\square$

Next, we examine different parameter as sufficient condition for 1-factors.

Connectivity Condition:

Let us start with connectivity, the first parameter one usually looks at, as an existence condition of 1-factors. The first significant result in this direction, actually the first on 1-factors in graphs, was obtained by Petersen (1891), who was particularly interested in 3-regular graphs.

Theorem 1.4.1 (Petersen [473]). *Let G be a 2-edge-connected 3-regular graph. Then G has a 1-factor.*

Proof. Let S be any nonempty proper subset of $V(G)$. Denote all the odd components of $G - S$ by $C_1, C_2, \dots, C_m$. From Proposition 1.4.1, we have $e_G(C_i, S) \geq 3$ for $1 \leq i \leq m$. Therefore

$$3|S| \geq e_G(C_1 \cup C_2 \cup \dots \cup C_m, S) \geq 3m.$$

Hence $|S| \leq m = o(G - S)$ or G has a 1-factor by 1-Factor Theorem. $\square$

The connectivity condition in Theorem 1.4.1 is best possible, i.e., there exist connected cubic graphs which have no 1-factors. An example is shown in Fig. 1.6.

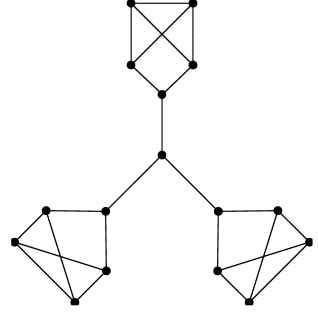


Fig. 1.6 A cubic graph without 1-factor.

This theorem can be generalized in several ways. Firstly, the 2-edge-connectivity condition can be replaced by “all the bridges of G are contained in the same path”; secondly, the theorem also holds for multigraphs as well; finally, of course, we can generalize it to general r -regular graphs. In fact, we prove a slightly stronger result by specifying a 1-factor to pass a given edge or to avoid other edges.

Theorem 1.4.2. *Let $r \geq 2$, and G be an $(r - 1)$ -edge-connected r -regular graph.*

- (a) *If $|V(G)| \equiv 0 \pmod{2}$, then, for every edge e of G , G has a 1-factor containing e . In particular, G has a 1-factor;*
- (b) *if $|V(G)| \equiv 1 \pmod{2}$, then, for every vertex v , $G - v$ has a 1-factor. That is, G is factor-critical.*

Proof. (a) Suppose, to the contrary, that there exists an edge $e = xy$ such that no 1-factor containing e . Then $G - \{x, y\}$ has no 1-factors. From 1-Factor Theorem, there exists $S' \subseteq V(G) - \{x, y\}$ such that $o(G - \{x, y\} - S') > |S'|$. Let $S = S' \cup \{x, y\}$. Then $o(G - S) \geq |S'| + 2 = |S|$ by parity. Let $C_1, C_2, \dots, C_m$ be the odd components of $G - S$, where $m = o(G - S)$. Then, by Proposition 1.4.1, we have $e_G(C_i, S) \geq r$. Hence

$$|\nabla(S)| \leq r|S| - 2 \leq rm - 2 \quad \text{since } e = xy \text{ is in } G[S]$$

and

$$|\nabla(S)| \geq e_G(C_1 \cup C_2 \cup \dots \cup C_m, S) \geq rm,$$

a contradiction. Hence, for every edge e of G , G has a 1-factor containing e .

(b) Suppose, to the contrary, that G is not factor-critical. Then, by Theorem 6.1.4 (in Section 6.1), there exists $\emptyset \neq S \subseteq V(G)$ such that

$o(G - S) > |S| - 1$ or $o(G - S) \geq |S| + 1$ by parity. Let $C_1, C_2, \dots, C_m$ be the odd components of $G - S$, where $m = o(G - S)$. Then, by Proposition 1.4.1, we have $e_G(C_i, S) \geq r$. Hence

$$r|S| \geq |\nabla(S)| \geq e_G(C_1 \cup C_2 \cup \dots \cup C_m, S) \geq rm \geq r(|S| + 1)$$

a contradiction. Hence G is factor-critical. $\square$

The edge-connectivity condition in Theorem 1.4.2 is best possible, i.e., there exist infinitely many $(r - 2)$ -edge-connected r -regular graphs of even order that have no 1-factors. The above theorem holds for multigraphs as well. Note that in part (b) r must be an even integer.

A classic result due to Plesník (1974) showed that there exist 1-factors to avoid a set of edges in regular graphs with edge-connectivity condition. The proof is from [14].

Theorem 1.4.3 (Plesník [477]). *Let $r \geq 2$ be an even integer, and G be an $(r - 1)$ -edge-connected r -regular graph of even order. Then for any $r - 1$ edges $e_1, e_2, \dots, e_{r-1}$ of G , G has a 1-factor excluding $\{e_1, e_2, \dots, e_{r-1}\}$.*

Proof. Let $H = G - \{e_1, \dots, e_{r-1}\}$. We show that H has a 1-factor. For any vertex set $S \subseteq V(H) = V(G)$, let $C_1, C_2, \dots, C_m$ be the odd components of $H - S$, where $m = o(H - S)$. Then by the same argument as in the proof of Proposition 1.4.1, we have

$$e_G(C_i, V(G) - V(C_i)) \geq r - 1 \quad \text{and} \quad e_G(C_i, V(G) - V(C_i)) \equiv r \pmod{2}.$$

Hence

$$r \leq e_G(C_i, V(G) - V(C_i)) = e_G(C_i, S) + e_G(C_i, V(G) - S - V(C_i)).$$

Since C_i and C_j ($i \neq j$) are components of $H - S$, if there is an edge e between C_i and C_j , then $e \in \{e_1, \dots, e_{r-1}\}$. That is, $E_G(C_i, V(G) - S - V(C_i)) \subseteq \{e_1, \dots, e_{r-1}\}$ and each of $\{e_1, \dots, e_{r-1}\}$ appears in at most two such sets $E_G(C_i, V(G) - S - V(C_i))$, and so it yields

$$\sum_{i=1}^m e_G(C_i, V(G) - S - V(C_i)) \leq 2|\{e_1, e_2, \dots, e_{r-1}\}| = 2(r - 1).$$

Therefore

$$\begin{aligned} rm &\leq \sum_{i=1}^m (e_G(C_i, S) + e_G(C_i, V(G) - S - V(C_i))) \\ &\leq \sum_{x \in S} d_G(x) + \sum_{i=1}^m e_G(C_i, V(G) - S - V(C_i)) \\ &\leq r|S| + 2(r - 1). \end{aligned}$$

Hence $m \leq |S| + 2(r-1)/r < |S| + 2$, which implies $m \leq |S|$ by parity. That is, $o(H - S) \leq |S|$ or H has a 1-factor. $\square$

Note that from Theorem 1.4.3, we can give a short proof to Theorem 1.4.2 (a): for a given edge $e = xy$, let the set of other $r-1$ edges incident to the vertex x be X , then there exists a 1-factor F excluding X and thus F is a 1-factor containing e .

Also note that the above result holds for multigraphs as well.

This result is best possible in the sense that there exist infinitely many $(r-2)$ -edge-connected r -regular graphs of even order that have no 1-factors.

Claw-free Condition:

A graph containing no $K_{1,3}$ as an induced subgraph is said to be *claw-free*. Claw-free graphs have been a popular condition to study for many graphic parameters. An important reason is that under claw-free condition many graph theory problems become significantly simpler. For instance, to determine a maximum independent set in arbitrary graphs is an NP-hard problem but becomes a P-problem if restricting to claw-free graphs.

Now we consider 1-factors in claw-free graphs.

Theorem 1.4.4 (Sumner [535]).

- (a) *If G is a connected claw-free graph of even order, then G has a 1-factor;*
- (b) *if a k -connected graph ($k \geq 2$) of even order has no induced subgraphs isomorphic to $K_{1,k+1}$, then G has a 1-factor.*

Proof. (a) Use induction on $|G|$. Let $P = v_1, v_2, \dots, v_k$ be a longest path of G . Then $k \geq 3$. It follows $N_G(v_1) \subseteq V(P) - \{v_1\}$.

Claim. $G - \{v_1, v_2\}$ is connected.

If $d_G(v_2) = 2$, then $G - \{v_1, v_2\}$ is connected since $N_G(v_1) \subseteq V(P) - \{v_1\}$. If $d_G(v_2) \geq 3$. For every $x \in N_G(v_2) - V(P)$, xv_3 must be an edge of G since, otherwise v_1x or v_1v_3 is an edge of G as the vertices x, v_1, v_2, v_3 do not induce a claw. However, this implies that G contains a path longer than P , a contradiction.

Therefore $G - \{v_1, v_2\}$ is connected, and of course, is claw-free, and has even order. By the induction hypothesis, $G - \{v_1, v_2\}$ has a 1-factor and so does G .

(b) Suppose, to the contrary, that G has no 1-factors. By 1-Factor Theorem, there exists a subset $S \subseteq V(G)$ such that $o(G - S) \geq |S| + 2$ by parity. Assume that S is a minimal such set and let $|S| = m$. Denote the odd components of $G - S$ by $O_1, O_2, \dots, O_t$. Thus $t \geq m + 2$. Moreover, $|\{v \in S \mid v \text{ is adjacent to a vertex of } O_i\}| \geq k$, for each $i = 1, 2, \dots, t$, and so there are at least k edges from O_i to S having distinct end-vertices in S . Choosing such a collection of k edges for each i gives a collection of

$kt \geq k(m+2)$ edges and hence there exists some vertex $x \in S$ incident with at least $k+1$ of these edges. But then x is the center of an induced $K_{1,k+1}$, a contradiction. $\square$

The above theorem can be generalized in several ways. Here we mention one by Ryjáček [508] and others are due to Nebešký [442]. We say that a graph G is *almost claw-free* if the centers of induced claws in G are independent and their neighborhoods are 2-dominated. It can be showed that every almost claw-free graph is $K_{1,5}$ -free.

Theorem 1.4.5 (Ryjáček [508]). *Every connected almost claw-free graph of even order has a 1-factor.*

Given a graph G , if it does not have 1-factors, by 1-Factor Theorem, there exists a vertex set S_0 such that $o(G - S_0) > |S_0|$ and $o(G - S_0) - |S_0| = \max_{S \subseteq V(G)} o(G - S) - |S|$ (Note that this set is called a *barrier set*). Sumner [537] showed that there exists a barrier set S_0 , each vertex of S_0 is a claw center. In fact, a minimal barrier set is such a set. With this remark, Tutte's 1-Factor Theorem can be restated in the following form.

Theorem 1.4.6 (Sumner [537]). *A graph G has 1-factor if and only if for any set S consisting of claw centers,*

$$o(G - S) \leq |S|.$$

Neighborhood Condition:

Next we discuss the existence of 1-factors in terms of neighborhood condition.

Theorem 1.4.7 (Win [586]). *Let G be a graph of even order n . If for any two nonadjacent vertices x and y ,*

$$d_G(x) + d_G(y) \geq n + 1,$$

then G has a hamiltonian cycle disjoint from a 1-factor. In particular, G has three disjoint 1-factors.

Theorem 1.4.8 (Anderson [54]). *Let G be a graph of even order n . If, for any subset $S \subseteq V(G)$ with $|S| \leq \frac{3n}{4}$, we have $|N_G(S)| \geq \frac{4}{3}|S|$, then G has a 1-factor.*

Proof. Suppose, to the contrary, that G has no 1-factors. By 1-Factor Theorem, there exists a vertex-subset $S_0 \subseteq V(G)$ such that $o(G - S_0) > |S_0|$.

Case 1. $|S_0| \geq \frac{n}{4}$.

Set $S = V(G) - S_0$. Then $|S| \leq \frac{3}{4}n$ and thus $|N_G(S)| \geq \frac{4}{3}|S|$. That is,

$$\frac{4}{3}(n - |S_0|) \leq |N_G(G - S_0)| \leq n - i(G - S_0),$$

i.e.,

$$i(G - S_0) \leq \frac{4}{3}|S_0| - \frac{n}{3}.$$

Thus

$$\begin{aligned} n &\geq |S_0| + i(G - S_0) + 3(o(G - S_0) - i(G - S_0)) \\ &> |S_0| + 3|S_0| - \frac{8}{3}|S_0| + \frac{2}{3}n. \end{aligned}$$

Hence $|S_0| < \frac{n}{4}$, a contradiction.

Case 2. $|S_0| < \frac{1}{4}n$.

Set $S = V(G) - S_0$. Then $|S| > \frac{3}{4}n$. Let S' be any subset of S with $|S'| = \frac{3}{4}n$. Then $|N_G(S')| \geq \frac{4}{3}|S'| = n$ and so $V(G) \subseteq N_G(S)$. Thus the vertices of S are adjacent to all the vertices of G . Consequently, there are no singleton components in $G - S_0$ and thus there are at least $|S_0| + 2$ odd components with at least 3 vertices in $G - S_0$. Consider all but one of these components. There are h vertices in their union, where $h \geq 3|S_0| + 3$. But they can be adjacent to at most $h + |S_0| \leq h + \frac{1}{3}(h - 3) < \frac{4}{3}h$ vertices. This contradiction completes the proof. $\square$

Note that the value $\frac{4}{3}$ in the above theorem is best possible. To see this, consider the graph $G = (r+2)K_3 \vee (rK_1)$. Let $S = V(rK_1)$. Then $o(G - S) = r + 2 > |S| = r$ and thus G has no 1-factors. Let $S' = V((r+1)K_3)$. The $|N_G(S')| = 3(r+1) + r$. Since $(4r+3)/(3r+3)$ can be made as close to $\frac{4}{3}$ as we like, it follows that the value $\frac{4}{3}$ is best possible.

Degree Sequences:

Let F be a 1-factor of a graph G . Then $G - F$ is a subgraph with the degree one less than that in G for each vertex, i.e., $d_{G-F}(x) = d_G(x) - 1$ for any $x \in V(G)$. This obvious fact turns out to be also sufficient for the existence of 1-factors in a graph. The next theorem gives a condition for the existence of 1-factors in terms of realization of degree sequences.

Theorem 1.4.9 (Kundu [329]; Lovász [405]). *Let G be a graph of even order with degree sequence $d_1, d_2, \dots, d_n$. If $d_1 - 1, d_2 - 1, \dots, d_n - 1$ is a degree sequence of some graph, then G has a 1-factor.*

Proof. Let $d_1, d_2, \dots, d_n$ be the degree sequence of G and $d_1 - 1, d_2 - 1, \dots, d_n - 1$ the degree sequence of another graph G' with $V = V(G) = V(G')$. Choose G' and G so that the symmetric difference of $E(G)$ and

$E(G')$, $E(G) \triangle E(G')$, is minimal. To show that G has a 1-factor, we need only to show that $G' \subseteq G$.

Suppose that $G' \not\subseteq G$, i.e., $E(G') \not\subseteq E(G)$. Let x_0 be a vertex which is incident with the most number, say r , of edges in $E(G') - E(G)$. Since $d_G(x_0) = d_{G'}(x_0) + 1$, the definition of symmetric difference implies that there are $r + 1$ edges in $E(G) - E(G')$ incident with x_0 . Due to the choice of x_0 and $E(G') \not\subseteq E(G)$, there exists a vertex $z \in V$ such that $zx_0 \in E(G') - E(G)$ and then we pick up another vertex $w \in V$ with $zw \in E(G) - E(G')$. Note that such a w exists since $d_G(z) = d_{G'}(z) + 1$.

Claim 1. For any $y \in V - \{x_0, w\}$ with $x_0y \in E(G) - E(G')$, then $wy \in E(G) - E(G')$.

Otherwise, $wy \notin E(G)$ or $wy \in E(G')$. If $wy \notin E(G)$, we replace the edges x_0y and wz by the edges x_0z and wy in G . The resulting graph has the same degree sequence, but with smaller symmetric difference with respect to G' , a contradiction to the minimality. Similarly, if $wy \in E(G')$, then replacing the edges x_0z and wy by the edges x_0y and wz in G' yields a contradiction, again.

From Claim 1, for every edge $yx_0 \in E(G) - E(G')$, we have the edge $yw \in E(G) - E(G')$ ($y \neq x_0, w$). In addition, there is also the edge wz . Note that the edge x_0w may or may not be an edge of $E(G) - E(G')$, but it contributes the same number to $d_G(x_0)$ and $d_G(w)$ in any case. Thus, we see that the number of edges in $E(G) - E(G')$ incident with w is greater than the number of edges in $E(G) - E(G')$ incident with x_0 , a contradiction to the choice of x_0 . $\square$

The above theorem can also be rewritten as a sufficient and necessary condition for the realization of degree sequence in graphs with 1-factors.

Corollary 1.4.1. *Let $d_1, d_2, \dots, d_n$ be integers. Then $d_1, d_2, \dots, d_n$ is a degree sequence of a graph with a 1-factor if and only if n is even and both sequences $d_1, d_2, \dots, d_n$ and $d_1 - 1, d_2 - 1, \dots, d_n - 1$ are degree sequences of graphs.*

Induced Subgraph and 1-Factors:

The existence of 1-factors in subgraphs can also ensure the existence of 1-factors in the original graph. This fact is firstly proved by Sumner [535]. In fact, this idea turns out to be very fruitful and can be applied to study many properties related to matchings. We shall present several results about extendability in induced subgraphs in Section 6.2 later.

Theorem 1.4.10 (Sumner [535]). *If G is a connected graph of order $2n$ ($n \geq 2$) and k is an integer, $1 \leq k \leq n$, such that every connected induced subgraph of order $2k$ has a 1-factor, then G has a 1-factor.*

Proof. We use induction on n . It is easy to check the validity of the theorem for $n = 2, 3$. Suppose that it holds for some $n - 1$ ($n \geq 4$). Let G and k be given in the theorem. Clearly, we may assume that $k \leq n - 1$, since the case of $k = n$ is trivial.

It is easy to see that G does not contain two leaves adjacent to a same vertex z , otherwise these two leaves together with z can be extended to a connected induced subgraph of order $2k$ which has no 1-factors.

Claim 1. There exist two adjacent vertices $x, y \in V(G)$ such that $G - \{x, y\}$ is connected.

Without loss of generality, assume that G is not complete and thus G has diameter $d \geq 2$. Let $a, y \in V(G)$ be vertices of G which are a distance d apart and let $P = a \dots xy$ be a path of length d joining a and y . Suppose that $G' = G - \{x, y\}$ is not a connected graph. Let C be the component of G' containing a . Since P is a diameter of G , every vertex in $G' - C$ is adjacent to x . Thus, if $G' - C$ contains a nontrivial component B , then for any $b, c \in B$ which are adjacent, every vertex in $G - \{b, c\}$ is joined to x by a path, and hence $G - \{b, c\}$ is connected. Similarly, if there exists a vertex $e \in G' - C$ which is adjacent to y , then $G - \{e, y\}$ is connected. Hence we may suppose that $G' - C$ contains only isolated vertices and each of these vertices is adjacent only to x . Thus, every vertex in $G' - C$ is a leaf of G adjacent to x . And since G has no two leaves adjacent to a same vertex, $G' - C$ must consist of a single isolated vertex f . If y were a leaf of G , then f and y would be leaves adjacent to a same vertex. Therefore, we may assume that y is adjacent to some vertex of C , and so $G - \{x, y\}$ is connected.

From Claim 1, $G - \{x, y\}$ is connected, and from induction hypothesis, $G - \{x, y\}$ has a 1-factor F . Hence, $F \cup xy$ constitutes a 1-factor of G . $\square$

Theorem 1.4.11 (Sumner [535]). *If G is a connected graph of even order and every induced $K_{1,3}$ contains a bridge whose deletion results in two components of even order, then G has a 1-factor.*

Proof. Use induction on $|G|$. It is easy to check the theorem holds for small orders. Assume the theorem holds for graphs with fewer vertices than $|G|$.

If G has no induced $K_{1,3}$, then G has a 1-factor by Theorem 1.4.4 (a). If G does contain an induced $K_{1,3}$, let $e = ab$ be an edge of this $K_{1,3}$ such that the deletion of e results in a graph having two even components A and B with $a \in A$ and $b \in B$. Suppose A contains an induced $K_{1,3}$. Let f be an edge of this subgraph such that $G - f$ contains two components of even order. Since both end-vertices of f lie in A , $A - f$ consists of two components C and D . We may assume that $a \in D$. Then C is one of the components of $G - f$ and hence has even order (as must D also). Thus A satisfies the condition of the theorem and so must have a 1-factor by induction hypothesis. Similarly,

B contains a 1-factor which together with that in A produces a 1-factor in G . $\square$

Other Conditions:

Many other parameters, e.g., toughness, binding number, independent number, genus, etc., can be used as sufficient conditions for 1-factors. Below we include two such theorems as conclusion of this section.

Let G be a connected graph. The *toughness* of G was introduced by Chvátal [151] and defined as

$$t(G) = \min \left\{ \frac{|S|}{\omega(G-S)} \mid S \subseteq V(G), \omega(G-S) \geq 2 \right\},$$

if G is not a complete graph; $t(G) = \infty$ if G is a complete graph. If $t(G) \geq k$, then we call G *k-tough*.

Theorem 1.4.12 (Chvátal [151]). *Let G be a connected 1-tough graph of even order. Then G has a 1-factor.*

Proof. For any nonempty subset $S \subset V(G)$, if $\omega(G-S) \geq 2$, then

$$|S| \geq \frac{|S|}{t(G)} \geq \omega(G-S) \geq o(G-S),$$

or $o(G-S) \leq |S|$. If $\omega(G-S) = 1$, then $o(G-S) \leq \omega(G-S) = 1 \leq |S|$ since S is nonempty. When $S = \emptyset$, then $o(G-S) = o(G) = 0 = |S|$ as G is of even order. That is, $o(G-S) \leq |S|$ for any subset S and thus G has a 1-factor. $\square$

The last parameter to consider is genus. The *genus*, $\gamma^*(G)$, of a graph G is defined to be smallest genus (i.e., number of handles) of all surfaces on which G can be embedded.

Theorem 1.4.13 (Nishizeki [464]). *Let G be an n -connected graph of even order. If $\gamma^*(G) < n(n-2)/4$, then G has a 1-factor.*

1.5 Gallai-Edmonds Structure Theorem

In this section, we present a structural theorem about matchings in graphs, which is a refinement of Tutte's 1-Factor Theorem. It is also the foundation for Edmonds' famous polynomial algorithm of maximum matchings. This theorem is a powerful accomplishment and it gives much more deeper characterizations to matchings as a subgraph. However, it seems this result has

not reached its full potential yet. It is our hope that more applications and appreciation of this theorem are gained in the future.

The structure developed in the proof of Tutte 1-Factor Theorem (i.e., Theorem 1.3.1) is very handy in many proofs of factor related theorems. Typically, in order to prove the existence of 1-factors, one uses a contrapositive argument by assuming the graph with the given conditions does not have 1-factors. Without loss of generality, we may choose a *maximal* such graph H , i.e., H has no 1-factors but if any edge is added, the resulting graph does. Such graphs possess many useful properties, we state it in the next theorem.

Theorem 1.5.1 (see [412]). *A graph G of even order is maximal non-1-factorable if and only if it has the following structure:*

G consists of vertex-disjoint complete graphs $S, G_1, G_2, \dots, G_k$ such that $k = |S| + 2$, where $G_1, \dots, G_k$ are of odd order and every vertex of G_i is connected to every vertex of S .

Proof. (outline of the proof)

Let G be a maximal non-1-factorable graph of even order. Define $S = \{x \mid d_G(x) = |V(G)| - 1\}$. We complete the proof by showing

- (1) $o(G - S) = |S| + 2$ (use parity and maximality);
- (2) there are no even components in $G - S$;
- (3) each component of $G - S$ is a complete graph. □

Based on Tutte's 1-Factor Theorem, we can easily obtained formulas for the deficiency and the matching number of a graph.

Theorem 1.5.2 (Berge's Formula). *Let $\text{def}(G)$ denote the deficiency of a graph G . Then*

$$\text{def}(G) = \max_{S \subseteq V(G)} \{o(G - S) - |S|\}, \quad (1.6)$$

$$\mu(G) = \frac{1}{2} \left(\min_{S \subseteq V(G)} \{|V(G)| - o(G - S) + |S|\} \right). \quad (1.7)$$

Proof. Let $d = \text{def}(G)$. At first, we notice that a matching can cover at most $|V(G)| - d$ vertices.

On the other hand, by Tutte's 1-Factor Theorem, the graph $H = G \vee K_d$ has a 1-factor. Those edges of a 1-factor of H that belongs to G do not cover at most d vertices of G .

Since $\mu(G) = \frac{1}{2}(|V(G)| - \text{def}(G))$, (1.7) follows immediately. □

We defined a *barrier* as a set $S \subseteq V(G)$ such that $o(G - S) = |S| + \text{def}(G)$. So barriers are such sets which reach the maximum in Berge's Formula. If G has a 1-factor, then $\emptyset$ is a barrier. An example of barrier is given in Fig. 1.7.

From Tutte's 1-Factor Theorem, we see that a graph G of even order has no 1-factors if and only if there exists a subset S such that $G - S$ has at

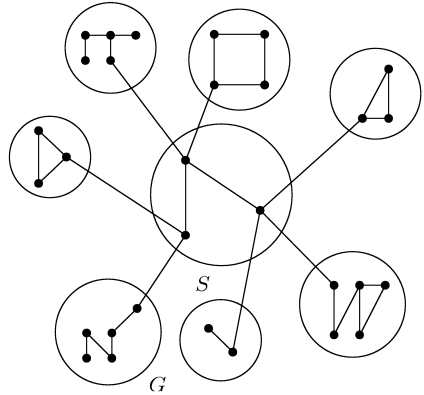


Fig. 1.7 A barrier S , where $\text{def}(G) = 2$ and $\mu(G) = 14$.

least $|S| + 2$ odd components. To further understand the structure of arbitrary graphs from the matching point of view, we refine Tutte's condition and introduce a canonical decomposition. This decomposition was studied by Gallai (1963) and Edmonds (1965), independently.

Recall that a graph G is *factor-critical* if $G - v$ has a 1-factor for every vertex v of G . It is easy to see that if G is factor-critical, then G is of odd order, connected and not a bipartite graph.

For a set $S \subseteq V(G)$, an *odd* (resp. *even*) S -*component* is a connected component of the graph $G - S$ of odd (resp. even) cardinality. The total number of vertices in those S -components which have no 1-factors is denoted by $\text{Def}(S)$, while the quantity $\text{def}_G(S) = o(G - S) - |S|$ is called the *deficiency* of S in G , or $\text{def}(S)$ in short. Directly from the definitions, we have the following two observations.

Observations:

- (1) Every matching in G leaves at least $\text{def}(S)$ vertices unsaturated;
- (2) if a matching M leaves exactly $\text{def}(S)$ vertices unsaturated, then M contains a 1-factor of each even S -component, a near-perfect matching of each odd S -component, and matches all vertices of S to vertices in distinct odd S -components.

For a vertex set S and a matching M , if M has exactly $\text{def}(S)$ vertices unsaturated (i.e., (2) holds), then S is called a *Tutte-Berge set*, while M is maximum by (1). The Berge's Formula (1.6) asserts that a Tutte-Berge set S always exists.

If $|S| + o(G - S) \geq 2$, we denote by $\langle G, S \rangle$ the bipartite graph obtained from G by deleting all the even S -components, contracting each odd S -component to a single vertex, and deleting the edges spanned by S . For a bipartite graph $H = (A, B)$, the set A satisfies Hall's condition (resp., with *positive surplus*)

if $|N_H(X)| \geq |X|$ (resp., $|N_H(X)| > |X|$) for every non-empty subset X of A . Therefore, Hall's Theorem asserts that H has a matching saturated every vertex of A if and only if A satisfies Hall's condition. Finally, we call S a *Gallai-Edmonds set* with respect to G , if S satisfies

- (a) all even S -components (if any) have a 1-factor;
- (b) all odd S -components (if any) are factor-critical;
- (c) if $S \neq \emptyset$, then S satisfies Hall's condition with positive surplus in $\langle G, S \rangle$.

Now we are ready to present and prove the famous Gallai-Edmonds Structure Theorem, which is a deep theorem for matchings. It provides a clear structure for graphs without 1-factors and it is the foundation of Edmonds' polynomial algorithm for finding a maximum matching in an arbitrary graph. It deserves more attention than its current status in authors' opinion. The proof used here is given by Kotlov [320] and it derives Gallai-Edmonds Structure Theorem from Hall's Theorem.

Theorem 1.5.3 (Gallai-Edmonds Structure Theorem). *Let G be a graph. Then*

- (i) *there exists a Gallai-Edmonds set S ;*
- (ii) *S is a Tutte-Berge set. That is, every maximum matching of G contains a near-perfect matching of each odd S -component, a 1-factor of each even S -component, and matches all vertices of S to vertices in distinct S -components;*
- (iii) *the underlying vertex set of all odd S -components is the set $D(G)$ of the vertices left unsaturated by at least one maximum matching of G , while S is the neighborhood of $D(G)$. In particular, G has a unique Gallai-Edmonds set.*

Proof. (i) Among all vertex subsets of G with maximum deficiency, let S have minimum $\text{Def}(S)$ and $|S| = s$. We use induction on $|G|$ to show that S is a Gallai-Edmonds set.

Suppose that C is an even S -component without 1-factors. For any fix $v \in V(C)$ and let $S' = S \cup \{v\}$. Then $\text{def}(S') \geq \text{def}(S)$ while $\text{Def}(S') < \text{Def}(S)$. This contradiction shows that S satisfies (a).

Suppose that v is a vertex in an odd S -component C such that $C - v$ has no 1-factors. By induction hypothesis, the graph $H = C - v$ has a Gallai-Edmonds set T . In particular, $\text{def}_H(T) \geq 2$. Set $S' = S \cup T \cup \{v\}$. Then $\text{def}(S') = o(G - S') - |S'| = (o(G - S) + o(H - T) - 1) - (s + |T| + 1) = \text{def}(S) + \text{def}_H(T) - 2 \geq \text{def}(S)$ while $\text{Def}(S') < \text{Def}(S)$, a contradiction. Thus, S satisfies (b).

Suppose that X is a smallest non-empty subset of S violating Hall's condition with positive surplus in $\langle G, S \rangle$. Set $S' = S - X$. If X consists of a single vertex with no neighbors in $\langle G, S \rangle$, then $\text{def}(S') > \text{def}(S)$, which is a contradiction. Otherwise, X satisfies Hall's condition in $\langle G, S \rangle$. By (b) and Hall's Theorem, X is contained in an even S' -component with a 1-factor (this

even component is spanned by X and the $|X|$ odd S -components neighboring with X in G). Consequently, $\text{def}(S') = \text{def}(S)$ and $\text{Def}(S') < \text{Def}(S)$. Thus, S satisfies (c).

Let now S be a Gallai-Edmonds set, and v a fixed vertex in an odd S -component C . By (c) and Hall's Theorem, S can be matched to vertices in s distinct odd S -components different from C . By (a) and (b), this matching can be extended by (near-)perfect matchings of the S -components to obtain a matching M avoiding v . Since M leaves $\text{def}(S)$ vertices unsaturated, it is maximum, and (ii) and (iii) follow immediately. $\square$

From Theorem 1.5.3, the Gallai-Edmonds set plays an important role in the structure of a graph. We can actually determine this Gallai-Edmonds set explicitly as follow: for a graph G , denote by $D(G)$ the set of all vertices in G which are *not* saturated by *at least* one maximum matching of G . Let $A(G)$ be the neighbor set of $D(G)$, i.e., the set of vertices in $V(G) - D(G)$ adjacent to at least one vertex in $D(G)$. Finally let $C(G) = V(G) - D(G) - A(G)$. Clearly, this partition is well-defined for every graph and does not rely on the choices of maximum matchings. In fact, the set $A(G)$ is our Gallai-Edmonds set. The partition $(D(G), A(G), C(G))$ is called a *canonical decomposition*.

This partition of $V(G)$ is also a refinement of Tutte's 1-factor Theorem, $A(G)$ can be thought as a Tutte-set S (which is a barrier), $D(G)$ is the union of the odd components of $G - S$ and $C(G)$ is the union of the even components of $G - S$. As an example, Fig. 1.8 shows the canonical decomposition $(D(G), A(G), C(G))$.

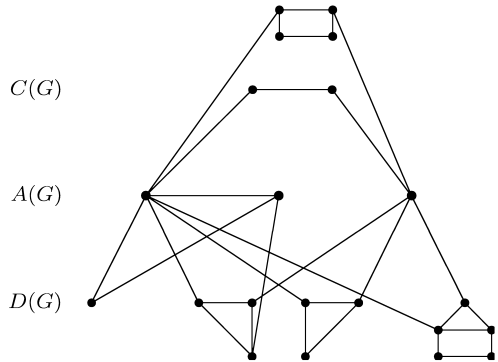


Fig. 1.8 The canonical decomposition $(D(G), A(G), C(G))$.

With the canonical decomposition $(D(G), A(G), C(G))$, we can restate Theorem 1.5.3 in the following form, which is the usual formation in most literatures.

Theorem 1.5.4 (Gallai-Edmonds Structure Theorem). *Let G be a graph and let $D(G)$, $C(G)$ and $A(G)$ be the vertex-partition defined above. Then*

- (i) *the components of the subgraph induced by $D(G)$ are factor-critical;*
- (ii) *the subgraph induced by $C(G)$ has a 1-factor;*
- (iii) *if M is any maximum matching of G , it contains a near-perfect matching of each component of $D(G)$, a 1-factor of each component of $C(G)$ and matches all vertices of $A(G)$ with vertices in distinct components of $D(G)$;*
- (iv) *the bipartite graph obtained from G by deleting the vertices of $C(G)$ and the edges spanned by $A(G)$ and by contracting each component of $D(G)$ to a single vertex has positive surplus (as viewed from $A(G)$);*
- (v) *the size of maximum matching is $\frac{1}{2}(|V(G)| - \omega(D(G)) + |A(G)|)$.*

Note: For a bipartite graph $G = (X, Y)$, the *surplus* of a set $S \subseteq X$ is defined by $|N(S)| - |S|$. The *surplus* of G is the minimum surplus of all non-empty subset of X .

Clearly, Theorem 1.5.4 provides more information than Tutte's 1-Factor Theorem. In fact, the latter can be easily derived from former. To see this, note that since $o(G - S) \leq |S|$ holds for all $S \subseteq V(G)$, in particular $o(G - A(G)) \leq |A(G)|$ and then substituting this into the formula in (v), we have $\frac{1}{2}(|V(G)| - \omega(D(G)) + |A(G)|) \geq \frac{1}{2}|V(G)|$. Thus G has a 1-factor. Similarly, let $S = A(G)$ and (v) yields equality of Berge's Formula.

In fact, the structure provided by Gallai-Edmonds Theorem has many other important consequences as well. Many of the results in this chapter are consequences of this powerful theorem.

Remark 1.5.1.

- (1) From the part (v), it is immediate that the deficiency of G , $def(G) = \omega(D(G)) - |A(G)|$; that is, it is the number of components of $D(G)$ left unmatched into $A(G)$ by an arbitrary maximum matching of G .
- (2) If G has no 1-factors, then every edge incident with a vertex of $D(G)$ lies in some maximum matching of G . To see this, let $u \in D(G)$ and uv be any edge incident to u . By the definition of $D(G)$, there is a maximum matching missing u . Hence by the maximality of M , there is an edge vw in M which covers v . But then $M - vw \cup uv$ is the desired matching. It follows from (iv) that no edge induced by $A(G)$ or connecting $A(G)$ to $C(G)$ belongs to any maximum matching. The edges in the subgraph induced by $C(G)$ show a much more complicated behavior in this respect and this problem will be discussed later in Chapter 2.
- (3) If G itself is a factor-critical graph, then we have a degenerate case of the canonical decomposition; namely, $D(G) = V(G)$, so $A(G) = C(G) = \emptyset$.
- (4) If G has a 1-factor, we have another degenerate case; namely, $D(G) = \emptyset$, so $A(G) = \emptyset$ and hence $C(G) = V(G)$.
- (5) If G is a bipartite graph with bipartition (X, Y) and if it has positive surplus (from X), then $A(G) = X$ and $D(G) = Y$. To see this, note that $\mu(G) = |X|$ and Hall's Theorem implies that $G - v$ has a maximum matching of X into $Y - v$ for every $v \in Y$. Hence $Y = D(G)$ and $X = A(G)$.

Thus the Gallai-Edmonds canonical decomposition is trivial in three cases: factor-critical graphs, graphs with 1-factors and bipartite graphs with positive surplus. On the other hand, it tells us that every graph can be built up from graphs of these three types.

Besides the properties mentioned in Theorem 1.5.4 and Remark 1.5.1, the sets $D(G)$, $C(G)$ and $A(G)$ have other interesting features as well. The next assertion was used as a lemma to prove Theorems 1.5.3 and 1.5.4 in several literatures. Even though we do not need it for the proof of Theorems 1.5.3, the assertion has its own merit to be mentioned.

Theorem 1.5.5 (Stability Lemma, see [412]). *Let G be a graph and let $D(G)$, $C(G)$ and $A(G)$ be defined above. Then*

- (i) *for any $u \in A(G)$, then $A(G - u) = A(G) - u$, $C(G - u) = C(G)$, and $D(G - u) = D(G)$;*
- (ii) *for any $u \in C(G)$, then $A(G - u) \supseteq A(G)$, $C(G - u) \subseteq C(G) - u$, and $D(G - u) \supseteq D(G)$;*
- (iii) *for any $u \in D(G)$, then $A(G - u) \subseteq A(G)$, $C(G - u) \supseteq C(G)$, and $D(G - u) \subseteq D(G) - u$.*

Proof. We only give the detailed proof of part (i) here. For parts (ii) and (iii), it can be proved similarly (see [412]).

(i) From the definitions of $D(G)$, $C(G)$ and $A(G)$, it is clear that we need only to show $D(G - u) = D(G)$.

Let M be a maximum matching of G and thus $\mu(G) = |M|$. Since $u \notin D(G)$, then M covers u . Hence $\mu(G - u) = \mu(G) - 1$. Moreover, since M is a maximum matching of G , $M - u$ is a maximum matching of $G - u$.

To see $D(G) \subseteq D(G - u)$, choose any $v \in D(G)$, let M_v be a maximum matching of G which misses v . Then $M_v - u$ is a maximum matching of $G - u$ and moreover, $M_v - u$ misses v too, so $v \in D(G - u)$. Thus $D(G) \subseteq D(G - u)$.

To see $D(G - u) \subseteq D(G)$, choose any $v \in D(G - u)$. Then there is a maximum matching M' of $G - u$ which misses v . Let w be any vertex in $D(G)$ adjacent to u in G and let M be a maximum matching of G misses w . If M misses v , then $v \in D(G)$ follows, so we assume that M covers v . Consider $M \cup M'$. By definition, M' avoids v . Thus the component of $M \cup M'$ covering v must be a path P starting at v with an edge of $M - M'$.

Suppose P ends with an edge of M' . Then $M \triangle P$ is a new matching M'' in G missing v . Moreover, $|M''| = |M|$, so M'' is a maximum matching and $v \in D(G)$. Otherwise, P ends with an edge of M . Let $M''' = M' \triangle P$. Then M''' is a matching and $|M'''| > |M|$. So $M''' \not\subseteq E(G - u)$ by the maximality of M' . Thus P must end at u . But now $M \triangle E(P \cup uw)$ is a maximum matching of G avoiding v and thus $v \in D(G)$. $\square$

For bipartite graphs, we have the following structural theorem.

Theorem 1.5.6 (see [412]). *Let $G = (U_1, U_2)$ be a connected bipartite graph, and let $C_1 = C(G) \cap U_1$ and $C_2 = C(G) \cap U_2$. Then the following statements hold*

- (i) $G[D(G)]$ is an independent set G ;
- (ii) $G[C(G)]$ has a 1-factor, and $|C_1| = |C_2|$;
- (iii) every maximum matching M in G consists of a 1-factor of $G[C(G)]$ and a matching in $G[A(G) \cup D(G)]$ saturating $A(G)$;
- (iv) both $A(G) \cap C_1$ and $A(G) \cap C_2$ are minimum vertex covers of G ;
- (v) both $D(G) \cap C_1$ and $D(G) \cap C_2$ are maximum independent subsets of G .

As an application of Gallai-Edmonds Structure Theorem, we prove the following result regarding the connectivity of allowed edges. More applications of Theorem 1.5.4 can be found in Section 5.1.

Theorem 1.5.7 (see [412]). *Let G be a connected graph with 1-factors. If $C(G - x) = \emptyset$ for all $x \in V(G)$, then all allowed edges form a connected spanning subgraph of G .*

Proof. Suppose, by the contrary, that there exists a partition of $V(G)$, say (V_1, V_2) where $V_1 \neq \emptyset \neq V_2$, such that all edges of $E(V_1, V_2)$ are not allowed, i.e., for any 1-factor M , we have $M \cap E(V_1, V_2) = \emptyset$. Thus M must match V_i onto itself and so $|V_i|$ is even ($i = 1, 2$).

Claim 1. There exists a subset S such that $|S| \geq 2$ and $o(G - S) = |S|$.

Let $v_1 v_2 \in E(V_1, V_2)$, where $v_1 \in V_1$ and $v_2 \in V_2$. Then $v_1 v_2$ is not allowed, that is, $G' = G - \{v_1, v_2\}$ has no 1-factors. From 1-Factor Theorem, there exists a subset $S' \subseteq V(G')$ such that $o(G' - S') > |S'|$ or $o(G' - S') \geq |S'| + 2$ by parity. Let $S = S' \cup \{v_1, v_2\}$. Then $o(G - S) = o(G' - S') \geq |S'| + 2 = |S|$. On the other hand, since G has 1-factors, we have $o(G - X) \leq |X|$ for any subset $X \subseteq V(G)$. Thus $o(G - S) = |S|$ or S is a barrier.

Without loss of generality, we assume that S is a *maximal* barrier.

Claim 2. Every edge $e = xy$ with $x \in S$ and $y \notin S$ is allowed in G .

Consider the graph $G - x$. Since $\text{def}(G) = 0$, so $\text{def}(G - x) = 1$. Because of $o(G - S) = |S|$, $x \in S$ and $\text{def}(G - x) = 1$, every maximum matching of $G - x$ saturates every vertex of $S - x$. By the definition of $D(G - x)$, so $S - x \subseteq A(G - x) \cup C(G - x) = A(G - x)$ as $C(G - x) = \emptyset$. On the other hand, by Stability Lemma (i.e., Theorem 1.5.5), $A(G - x) = A(G) - x$ and thus $A(G - x) \cup x = A(G)$ is a barrier in G . By the maximality of S , $A(G - x) \cup x \subseteq S$. Thus

$$S = A(G - x) \cup x. \quad (1.8)$$

Since $y \notin S$, by (1.8), $y \in D(G - x) \cup C(G - x) = D(G - x)$ as $C(G - x) = \emptyset$ again. So there exists a maximum matching M_x of $G - x$ missing y but saturating all other vertices of $G - x$. Then $M_x \cup \{xy\}$ is a 1-factor of G , i.e., xy is an allowed edge of G .

Claim 3. $G - S$ has no odd components joined to both $S \cup V_1$ and $S \cap V_2$.

Suppose, to the contrary, that H is such a component. Let uv be an edge with $u \in V(H)$ and $v \in S \cap V_1$. By Claim 2, there exists a 1-factor F of G containing uv . Thus $u \in V_1$ by the definition of V_1 .

Since $|V(G)|$ is even, we have $D(G - v) \neq \emptyset$ but $C(G - v) = \emptyset$. Let e_v be an edge of F which is incident to v . Then $F - e_v$ is a maximum matching of $G - v$. Moreover, $F - e_v$ saturates all vertices of $A(G - v)$ and has at most one edge joining each odd component of $D(G - v)$ (of which H is one) to $A(G - v)$. Recall $S = A(G - v) \cup v$, upon reinstalling v and e_v we have exactly one edge of F joining S to each component of $G - S$ and in particular, exactly one edge of F joining H to S . So $E(H) \cap F$ is a near-perfect matching of H . Thus $E(H - V_2) \cap F$ is a near-perfect matching of $H - V_2$, and so $|V(H) \cap V_1|$ is odd. Similarly, $|V(H) \cap V_2|$ is odd and hence $|V(H)|$ is even, a contradiction.

By Claims 2 and 3, all odd components of $G - S$ are joined to $S \cap V_1$ or $S \cap V_2$, but not both. For $i = 1, 2$, let a_i be the number of odd components of $G - S$ joined to $S \cap V_i$. Then $a_1 + a_2 = o(G - S) = |S|$. Without loss of generality, assume $a_1 \leq |S \cap V_1|$. Let $x \in S \cap V_2$. Then, from (1.8), $A(G - v)$ has a subset of $k = |S \cap V_1|$ vertices joined to at most k odd components of $(G - x) - A(G - x)$, which contradicts to the part (iv) of Gallai-Edmonds Structure Theorem.

The proof is complete. $\square$

Kelmans [313] studied the *induced* star packing problem as a natural generalization of the matching problem and obtained a Gallai-Edmonds type of structure theorem for induced star packing and also included Tutte's 1-Factor Theorem, Berge's Formula, the matching matroid theorem as special cases.

For hypergraphs, Štehlík [531] generalized the notions of matchings, matching-covered graphs and factor-critical graphs to hypergraphs and used these concepts to extend the Gallai-Edmonds Structure Theorem to hypergraphs,

Puleyblank [502] extended Theorem 1.5.4 to fractional matchings. We will provide more details in Section 8.1.

1.6 Number of 1-factors

Let $\Phi(G)$ denote the number of 1-factors in a graph G . One of focuses in the study of matching theory is to determine the function $\Phi(G)$. However, it seems very unlikely that a formula or an efficient algorithm for computing $\Phi(G)$ will ever be found. In particular, Valiant (1979) proved that the problem of determining $\Phi(G)$ is NP-hard, even when G is bipartite. This left very little

room for finding the exact value of $\Phi(G)$. Naturally, the next move is to find a lower bound for $\Phi(G)$.

Let's first present a few classical results in this direction.

Theorem 1.6.1 (Hall (1948)). *Let $G = (X, Y)$ be a bipartite graph with a 1-factor and $d(x) \geq k$ for every $x \in X$. Then $\Phi(G) \geq k!$.*

Theorem 1.6.2 (Beineke and Plummer (1967)). *Let G be a 2-connected graph with a 1-factor. Then $\Phi(G) \geq 2$.*

Theorem 1.6.3 (Zaks (1970)). *Let G be a k -connected graph with a 1-factor ($k \geq 2$). Then $\Phi(G) \geq k!!$. In particular, $\Phi(K_n) = (n-1)!! = (n-1)(n-3)\dots 1$. The bounds are sharp when k is odd.*

Theorem 1.6.4. *Let G be a connected graph of order n with a unique 1-factor. Then*

- (a) G has a bridge belonging to the 1-factor;
- (b) G contains a vertex of degree $\leq \lfloor \log_2(n+1) \rfloor$; and
- (c) $|E(G)| \leq \frac{1}{4}n^2$.

Note that, in the above theorem, (a) is from Kotzig (1959); (b) is due to Lovász (1972) and (c) is from Heteyi (unpublished manuscript, see [401]).

Theorem 1.6.5 (Lovász [401]). *Let G be a k -connected graph with a 1-factor and G is not bicritical. Then $\Phi(G) \geq k!$*

An immediate consequence of Theorem 1.6.5 is that a k -connected graph G with 1-factors has $\Phi(G) \geq k!$ or G is bicritical. Bicritical graphs are a family of important graphs, and it plays a unreplaceable role in the study of matchings, for example, it can be used to develop a decomposition theory for elementary graphs (ref. Section 5.2) and it can also be used to determine the dimension of 1-factors lattice (ref. Section 5.4).

From the definition of bicriticality, it seems that this family of graphs are very rich in 1-factors. However, bicritical graphs is somewhat counterintuitive. In fact, bicritical graphs contain much less 1-factors than other graphs and is more difficult to bound $\Phi(G)$ than most other graphs, as shown in the next theorem. This theorem has a weaker condition, 2-connected and $\delta(G) \geq k$, than Theorem 1.6.5.

Theorem 1.6.6 (Mader (1976)). *Let G be a 2-connected graph with $\delta(G) \geq k$ and have a 1-factor. Then either*

- (a) *there are at least $k!$ 1-factors in G (i.e., $\Phi(G) \geq k!$), or*
- (b) *G is bicritical. In this case, $\Phi(G) \geq |V(G)|/2 + 1$.*

The investigation of lower bound for general graphs is still ongoing. On the other hand, the lower bounds for some special families of graphs are also researched. For example, Lovász and Plummer [409] obtained the best

possible lower bound $\Phi(H)$ for Halin graphs H (see Theorem 5.3.2 for the details).

It has been shown that one can bound $\Phi(G)$ below by a certain matrix function called Pfaffian. In the case when G is planar, the Pfaffian matrices can be used to exactly compute $\Phi(G)$ in polynomial time. (The definition of Pfaffian orientation can be found in Section 5.4.)

Now we turn to the problem of finding a lower bound for the number of near-perfect matchings in a factor-critical graphs. The following result was obtained by Pulleyblank (1973) and he used linear programming technique to prove it. Here we present a short proof from [412].

Theorem 1.6.7 (Pulleyblank [500]). *Every 2-connected factor-critical graph G contains at least $|E(G)|$ near-perfect matchings.*

Proof. Since G is a 2-connected factor-critical graph, from Theorem 5.2.5, so G can be expressed as $G = P_0 + P_1 + P_2 + \cdots + P_m$, where P_0 is an odd cycle and each P_r ($1 \leq r \leq m$) is an open path of odd length joining two different vertices of $P_0 + P_1 + \cdots + P_{r-1}$. Denote that $G_r = P_0 + P_1 + \cdots + P_r$. Then G_r is factor-critical.

We use induction on m . If $m = 0$, the result clearly holds. Suppose the result holds for $r \geq 0$. We consider $G_{r+1} = G_r + P_{r+1}$, which is a 2-connected factor-critical graph. Let $P_{r+1} = xu_1 \dots u_{l-1}y$, where $l = 1$ or $l \geq 3$.

If F is any near-perfect matching of G_r , then $F' = F \cup \{u_1u_2, \dots, u_{l-2}u_{l-1}\}$ is a near-perfect matching of G_{r+1} . By induction hypothesis, there are $|E(G_r)| = |E(G_{r+1})| - l$ near-perfect matchings in G_{r+1} .

Since G_r is factor-critical, for each u_i ($1 \leq i \leq l-1$) we can easily find a 1-factor F_{u_i} of $G_{r+1} - u_i$. So we obtain $l-1$ more near-perfect matchings of G_{r+1} . Note that each near-perfect matching F_{u_i} saturates only one of x and y with an edge of P_{r+1} . Next we construct the last near-perfect matching of G_{r+1} , which saturates both of x and y with the edges of P_{r+1} . Let F_x be a 1-factor of $G_r - x$. Then F_x must cover y with an edge $yz \in E(G_r)$. Therefore $F'' = (F_x - \{yz\}) \cup \{xu_1, u_2u_3, \dots, u_{l-1}y\}$ is a 1-factor of $G_{r+1} - z$ and thus the desired near-perfect matching of G_{r+1} .

We complete the induction. $\square$

Recently, Došlić (2008) used ear-decomposition theory to establish lower bounds on the number of perfect matchings in k -extendable graphs.

Theorem 1.6.8 (Došlić [164]). *Let G be a k -extendable graph of p vertices and q edges with maximum degree Δ , where $k \geq 1$. Then $\Phi(G) \geq \lceil \frac{(k+1)!}{4} (q - p - (k-1)(2\Delta-3) + 4) \rceil$.*

Now we divert our attention from determining the value of $\Phi(G)$ to study of the structure of graphs which are sensitive to $\Phi(G)$. The most results in this area are due to Lovász [401]. Reader may choose to read this part after

Chapter 5 (in particular, Section 5.1) since some terms and results are given in details there.

A graph G with a 1-factor is Φ -*maximal* if $\Phi(G \cup e) > \Phi(G)$ for any edge $e \notin E(G)$, i.e., the graph $G \cup e$ has more 1-factors than G . Note that a 1-extendable graph is Φ -maximal if and only if it is bicritical. Every graph with 1-factors is contained in a Φ -maximal graph because we can add edges to a graph as long as no new 1-factors are generated. Many important graph-theoretic properties are preserved by this procedure.

A subset $S \subseteq V(G)$ is called *extreme* if $\text{def}(G - S) = \text{def}(G) + |S|$. From the definition of extreme sets, it is easy exercises to see the following properties of extreme sets.

Lemma 1.6.1. *Let G be a graph. Then*

- (a) *every subset of an extreme set is extreme;*
- (b) *if S is an extreme set and $S \subseteq T$, then $T - S$ is an extreme set too.*

Below, we characterize Φ -maximal graphs.

Theorem 1.6.9. *Let G be any graph with 1-factors. Then G is Φ -maximal if and only every extreme set in G induces a complete graph, all edges of which are forbidden in G .*

Proof. Assume that G is Φ -maximal. Let X be an extreme set in G and $\{x, y\} \subseteq X$. By Lemma 1.6.1, $\{x, y\}$ is an extreme set of G .

Suppose x and y are not adjacent in G . Then $\text{def}((G \cup xy) - x - y) = \text{def}(G - x - y) = \text{def}(G) + 2$ since $\{x, y\}$ is extreme in G . But $\text{def}(G \cup xy) = \text{def}(G) = 0$, so we have $\text{def}((G \cup xy) - x - y) = \text{def}(G \cup xy) + 2$ and thus $\{x, y\}$ is extreme in $G \cup xy$. Therefore, xy is forbidden in $G \cup xy$ and so G is not Φ -maximal, a contradiction. Then we must have $xy \in E(G)$ and thus xy is forbidden in G .

Conversely, suppose G is not Φ -maximal and thus there are two vertices $x, y \in V(G)$ with $xy \notin E(G)$, such that G and $G \cup xy$ have the same 1-factors. Thus xy is forbidden in $G \cup xy$ and $\{x, y\}$ is extreme in $G \cup xy$. So $\text{def}(G - x - y) = \text{def}((G \cup xy) - x - y) = \text{def}(G \cup xy) + 2 = 0 + 2 = \text{def}(G) + 2$ and thus $\{x, y\}$ is extreme in G . But this contradicts the hypothesis that xy must be a forbidden edge of G . $\square$

In addition, if a Φ -maximal graph is also elementary (i.e., allowed edges form a connected subgraph), then Theorem 1.6.9 yields the following.

Theorem 1.6.10. *If G is a Φ -maximal elementary graph, then the forbidden edges of G constitute vertex-disjoint complete subgraphs induced by the classes of $\mathcal{P}(G)$.*

The next result shows us that the Construction of general elementary graphs discussed in Section 5.2 is “more unique” when graphs are Φ -maximal.

Theorem 1.6.11. *Let G be a Φ -maximal elementary graph and let G^* be the graph obtained by contracting each class S of $\mathcal{P}(G)$ to a vertex t_S (see an example in Fig. 1.9). Then*

- (a) *the block of G^* (i.e., the maximal 2-connected subgraph of G^*) are bicritical;*
- (b) *the number of blocks of G^* containing t_S is exactly $|S|$;*
- (c) *if we build G by iterating the construction of Theorem 5.1.9, then the initial bicritical graphs must be precisely the blocks of G^* , and*
- (d) *the elementary bipartite graphs used in the last steps of the Construction of G are uniquely determined in the sense that they must be of the form G'_S , for some $S \in \mathcal{P}(G)$.*

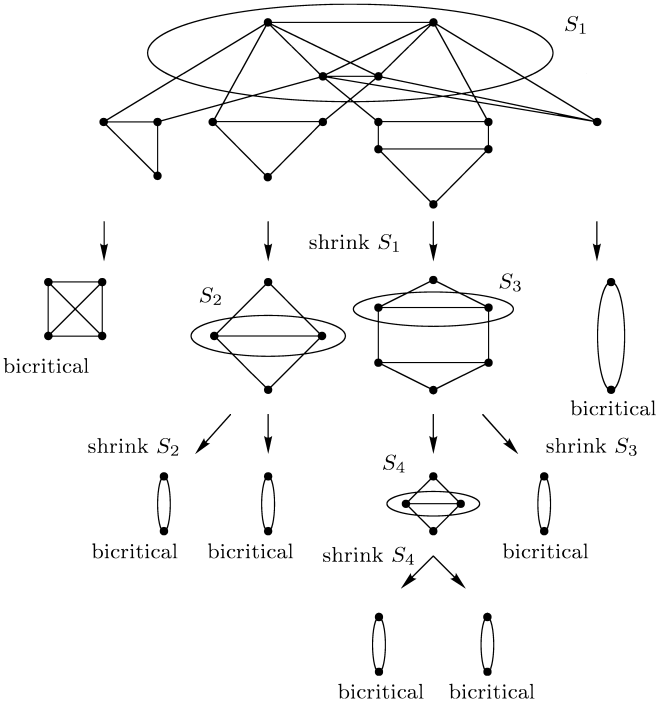


Fig. 1.9 Construction of a Φ -maximal elementary graph.

Now we proceed to the main result of this section – Cathedral Theorem – for all Φ -maximal non-elementary graphs. First, we introduce Cathedral Construction, which was studied by Kotzig (1960):

Cathedral Construction. Let G_0 be any Φ -maximal elementary graph. To each class $S \in \mathcal{P}(G_0)$ assign an (already constructed) Φ -maximal graph G_S or the empty set. For each $S \in \mathcal{P}(G_0)$ join every vertex of S to every vertex

of G_S . If G_S is not empty, we call the subgraph G_S the *tower* over S and call S the *foundation* of that tower. (Note that a tower and its foundation are vertex-disjoint.)

Now we can state Cathedral Theorem, but the proof is long and involved and we omit it.

Theorem 1.6.12 (Cathedral Theorem).

- (a) Every graph G , built up by iterating Cathedral Construction using smaller Φ -maximal graphs, is itself Φ -maximal;
- (b) the allowed edges of G are precisely those edges which are allowed in one of the elementary graphs used in one of the steps.
- (c) Conversely, if G is any Φ -maximal graph, it can be built up using Cathedral Construction starting with a Φ -maximal elementary graph G_0 and a collection of $|\mathcal{P}(G_0)|$ smaller Φ -maximal graphs (some perhaps empty) already constructed. The graph G_0 may be uniquely described as the subgraph of G induced by those vertices of G which, for each $x \in V(G)$, does not lie in $C(G - x)$.

An illustration of Cathedral structure is shown in Fig. 1.10.

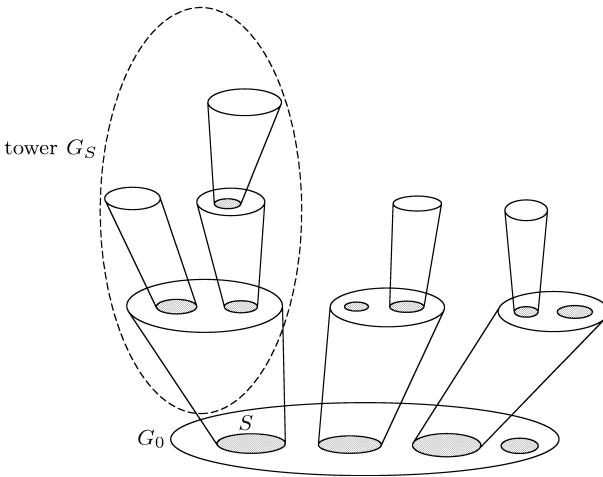


Fig. 1.10 A Cathedral construction.

Determining $\Phi(G)$ for a graph G is an extremely difficult task, even for a special class of graphs, e.g., bipartite graph. One of important tools in studying $\Phi(G)$ of *bipartite graphs* is to use the permanent of incident matrix of G . There is a direct correspondence between binary $n \times n$ matrices and balanced bipartite graphs of order $2n$. So to compute the exact value of $\Phi(G)$ is equivalent to find the permanent of corresponding matrices. Even

though to determine the permanent of matrices is again an NP-complete problem, however the equivalence between the permanent and $\Phi(G)$ leads to the discovery of a good non-trivial lower bound for $\Phi(G)$ when G is bipartite and regular. The lower bound and upper bound are obtained by Schrijver and Brègman, respectively.

Theorem 1.6.13 (Schrijver (1998); Brègman (1973)). *Let G be a k -regular bipartite graph on $2n$ vertices. Then*

$$\left(\frac{(k-1)^{k-1}}{k^{k-2}} \right)^n \leq \Phi(G) \leq (k!)^{n/k}.$$

As a conclusion of this section, as well as the chapter, we pose a conjecture on $\Phi(G)$ by Lovász and Plummer (1989).

Conjecture 1.6.1 (Lovász and Plummer [413]). *Let G be a 2-edge-connected cubic graph G . Then there exists a constant $c > 1$ such that $\Phi(G) \geq c^n$.*

There are some partial results known for this conjecture. For example, Voorhoeve (1979) showed that if G is a cubic bipartite graph on $2n$ vertices, then $\Phi(G) \geq (4/3)^n$.

Chapter 2

Degree Constraint Factors

Factor problems are naturally divided into two groups, degree constraint factors and component factors, based on the properties of subgraphs. Of course, these two groups are not entirely disjoint. For instance, 1-factors can be considered as a spanning subgraph F satisfying degree condition: $d_F(x) = 1$ for every vertex x ; or it can be also thought as a spanning subgraph with each component isomorphic to K_2 . Factor problems may also overlap with other graph theory problems, for example, Hamilton cycle problem and subgraph packing problem.

In this chapter, we focus on degree constraint factors, which include many well-known factors, such as 1-factors, k -factors and $[a, b]$ -factors, etc. In Section 2.1, we present the characterizations of various factors. The proofs of these characterizations tend to be lengthy, however they are already well documented by other books (e.g., [14], [412]). Without compromising the completeness of the book, we do not provide proofs for these characterizations in the most cases. In Section 2.2, we mainly investigate factors in bipartite graphs and the graphs alike (e.g., graphs with the odd-cycle property). In Section 2.3, we study the properties of factors, such as continuity, inclusion and exclusion, etc. In last section, we consider the most general degree constraint factors, L -factors or list-factors. Unlike the other degree constraint factors which have polynomial algorithms to determine it, L -factor problems are almost always in NP-complete class. It has close relations with many combinatorial optimization problems and has many applications. However, it is still an area under-developed in factor theory.

2.1 Characterizations of factors

In this section, we attempt to summary all known characterizations for various factors, but without proofs in most cases. The functions used in this

chapter are all defined on vertex set $V(G)$ with nonnegative integer-values. For convenience, denote $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$ and thus a function is denoted by $f : V(G) \rightarrow \mathbb{Z}^+$. Note that all characterizations in this section also hold for the graphs with *multiple edges*, even though we state the theorems for simple graphs only.

Let H be a spanning subgraph of a graph G . If $d_H(x) = k$ for all $x \in V(G)$, then H is called a k -factor. Clearly, a 1-factor is a perfect matching and a 2-factor is a collection of cycles.

There are many generalizations to k -factors, such as f -factors, (g, f) -factors, $[a, b]$ -factors and L -factors. The first of such generalizations is f -factors introduced by Tutte [561] in 1952. Let f be an integer-valued function defined on $V(G)$. An f -factor of a graph G is a subgraph H such that $d_H(x) = f(x)$ for all $x \in V(G)$.

The more general version of degree constraint factors, (g, f) -factors, was investigated by Lovász [399]. Let g and f be integer-valued functions defined on $V(G)$, i.e., $g, f : V(G) \rightarrow \mathbb{Z}$. A spanning subgraph H is called a (g, f) -factor if $g(x) \leq d_H(x) \leq f(x)$ for all $x \in V(H)$. In particular, if $g(x) = a$ and $f(x) = b$ for all $x \in v(G)$, where a and b are two fixed integers, then the (g, f) -factor is called an $[a, b]$ -factor. Moreover, if $g(x) \equiv f(x) \pmod{2}$ for all $x \in V(G)$, a (g, f) -factor F with $d_F(x) \equiv f(x) \pmod{2}$ for all $x \in V(G)$ is called *parity* (g, f) -factor.

The characterizations for all factors introduced above are all along a same line, which involve to compute the quantity $\delta(S, T)$ for all possible disjoint vertex-subsets S and T . For different types of factors, $\delta(S, T)$ is defined slightly different. But to avoid to introduce too many notions, we use the same notion $\delta(S, T)$ for all factors. This should cause no confusion since each theorem usually only deal with one type of factors, so we use the respective $\delta(S, T)$.

Let S and T be two disjoint subsets of $V(G)$. Recall that the set of edges joining S and T in G is denoted by $E_G(S, T)$ and let $e_G(S, T) = |E_G(S, T)|$. For a subset $X \subseteq V(G)$, we write

$$f(X) = \sum_{x \in X} f(x).$$

For k -factors, let C be a component of $G - (S \cup T)$. We call C a k -odd (or k -even) component if the parity of $k|C| + e_G(T, V(C))$ is odd (or even). Just like Tutte's 1-Factor Theorem, the number of k -odd components plays a crucial role in the characterization of k -factors. We denote the number of k -odd components of $G - (S \cup T)$ by $q_G(S, T)$. For other factors, we can define the odd/even components and $q_G(S, T)$ in similar manners. If there is no confusion arisen, we simply refer k -odd components (and other types of odd components) as odd components in the coming sections. However, for

the purpose of reference, we still give the definition of $q_G(S, T)$ for each type of factors in their respective theorems.

Now we are ready to define

$$\delta_G(S, T) = \begin{cases} k|S| + \sum_{x \in T} d_G(x) - k|T| - e_G(S, T) - q_G(S, T) & \text{for } k\text{-factor;} \\ f(S) + \sum_{x \in T} d_G(x) - f(T) - e_G(S, T) - q_G(S, T) & \text{for } f\text{-factor;} \\ f(S) + \sum_{x \in T} d_G(x) - g(T) - e_G(S, T) - q_G(S, T) & \text{for } (g, f)\text{-factor.} \end{cases}$$

Note that

$$\sum_{x \in T} d_{G-S}(x) = \sum_{x \in T} d_G(x) - e_G(S, T).$$

So $\delta_G(S, T)$ has other forms of expressions too.

Next we present the characterizations of different constraints factors. The following well-known k -Factor Theorem and f -Factor Theorem are due to Tutte [561] in 1952.

Theorem 2.1.1 (k -Factor Theorem). *Let G be a graph and k a positive integer. Then G has a k -factor if and only if, for all $S, T \subseteq V(G)$ with $S \cap T = \emptyset$,*

$$\delta_G(S, T) = k|S| + \sum_{x \in T} d_G(x) - k|T| - e_G(S, T) - q_G(S, T) \geq 0, \quad (2.1)$$

where $q_G(S, T)$ is the number of components C of $G - (S \cup T)$ such that $e(V(C), T) + k|C| \equiv 1 \pmod{2}$. Moreover, $\delta_G(S, T) \equiv k|V(G)| \pmod{2}$.

Theorem 2.1.2 (f -Factor Theorem). *Let G be a graph and $f : V(G) \rightarrow \mathbb{Z}^+$ be a function. Then G has an f -factor if and only if, for all $S, T \subseteq V(G)$ with $S \cap T = \emptyset$,*

$$\delta_G(S, T) = f(S) + \sum_{x \in T} d_G(x) - f(T) - e_G(S, T) - q_G(S, T) \geq 0, \quad (2.2)$$

where $q_G(S, T)$ is the number of components C of $G - (S \cup T)$ such that $e(V(C), T) + \sum_{x \in V(C)} f(x) \equiv 1 \pmod{2}$. Moreover, $\delta_G(S, T) \equiv \sum_{x \in V(G)} f(x) \pmod{2}$.

Remark 2.1.1. The original proof of f -Factor Theorem is lengthy and difficulty. However, there is shorter proof provided by Tutte [563], who constructed a new graph G^* from G and then proved that G has an f -factor if and only if G^* has a 1-factor. We describe this reduction method below.

Assume that $f(x) \leq d(x)$ for all $x \in V(G)$, otherwise G has too few edges at x to have an f -factor. Let $e(x) = d(x) - f(x)$, which is the excess degree at x and is nonnegative.

To construct G^* , replace each vertex x with a complete bipartite graph $K_{d(x), e(x)}$ having bipartition $A(x)$ of size $d(x)$ and $B(x)$ of size $e(x)$. For each $uv \in E(G)$, add an edge joining one vertex of $A(u)$ to one vertex of $A(v)$. Each vertex of $A(u)$ participates in one such edge.

We use the following example to illustrate the construction of G^* .

Example 2.1.1. Given a graph G and we construct the graph G^* in Fig. 2.1. The graph on the left is G with vertex label given by f , and G^* is on the right side. The bold edges in G^* form a 1-factor that corresponds to an f -factor of G .

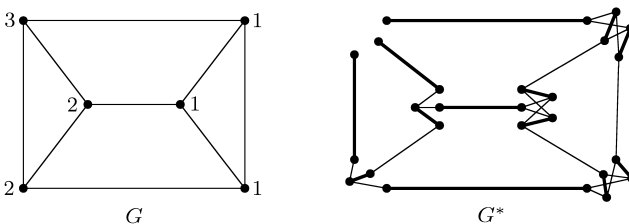


Fig. 2.1 Construction of G^* .

Next, we consider (g, f) -factors. Lovász [399] gave a criterion (so-called (g, f) -Factor Theorem) for the existence of (g, f) -factors in 1970. The (g, f) -Factor Theorem is much more general than k -Factor Theorem and f -Factor Theorem. It holds not only for general graphs (with loops and multiple edges) and also allows the values of functions g, f to be negative. The original proof is long and very involved. In 1981, Tutte [566] gave an elegant short proof to this theorem.

For any subgraph H , it may have some vertices violating the degree constraints, i.e., there exists vertex u so that $d_H(u) < g(u)$ or $d_H(u) > f(u)$. We can introduce a quantity to measure the “deviation” or “deficiency” of H ,

$$def(H) = \sum_{x \in V(H)} (\max\{0, g(x) - d_H(x)\} + \max\{0, d_H(x) - f(x)\}).$$

The *total deficiency* of a graph G is defined as

$$\text{def}(G) = \min_{H \subseteq G} \text{def}(H).$$

Theorem 2.1.3 ((g, f)-Factor Theorem, Lovász [399]). *Let G be a graph and $g, f : V(G) \rightarrow \mathbb{Z}$ such that $g(x) \leq f(x)$ for all $x \in V(G)$. Then G has a (g, f) -factor if and only if*

$$\delta_G(S, T) = \sum_{t \in T} (d_G(t) - g(t)) + \sum_{s \in S} f(s) - e_G(S, T) - q_G(S, T) \geq 0, \quad (2.3)$$

for all disjoint subsets S and T of $V(G)$, where $q_G(S, T)$ denotes the number of components C of $G - (S \cup T)$ such that $g(x) = f(x)$ for all $x \in V(C)$ and $e(V(C), T) + \sum_{x \in V(C)} f(x) \equiv 1 \pmod{2}$.

Moreover, G has a parity (g, f) -factor if and only if for all disjoint S and T ,

$$\tilde{\delta}(S, T) = \sum_{t \in T} (d_G(t) - g(t)) + \sum_{s \in S} f(s) - e_G(S, T) - \tilde{q}_G(S, T) \geq 0, \quad (2.4)$$

where $\tilde{q}_G(S, T)$ is the number of components C of $G - (S \cup T)$ such that $e(V(C), T) + \sum_{x \in V(C)} f(x) \equiv 1 \pmod{2}$.

Remark 2.1.2. In (g, f) -Factor Theorem, the two parameters $\delta_G(S, T)$ and $q_G(S, T)$ sometimes are written as $\delta_G(S, T; g, f)$ and $q_G(S, T; g, f)$, respectively, if more than one (g, f) -factors are considered so to distinguish the parameters for different factors; in other cases they are used as $\delta(S, T)$ and $q(S, T)$ for short hands if there is no confusion arisen. We adopt the same custom for other factors, i.e., k -factors, f -factors, and $[a, b]$ -factors.

Theorem 2.1.3 is not easy to use due to the difficulty of computing $q_G(S, T)$ (or $\tilde{q}_G(S, T)$). However, for some special cases, the characterization can be simplified and becomes a more practical criterion. Below, we consider the condition “ $g(x) < f(x)$ for all $x \in V$ ” (other such “nice” conditions include that “ G is bipartite” or “ G has odd-cycle property”). In this case, it does not involve to compute $q_G(S, T)$. The following theorem is proved and “rediscovered” by many researchers in different but equivalent forms, for example Lovász [404], Heinrich *et al.* [239], Kano and Saito [288], Addario-Berry *et al.* [3].

Theorem 2.1.4. *Let G be a graph and $g, f : V(G) \rightarrow \mathbb{Z}^+$ such that $g(x) < f(x)$ for all $x \in V$. Then G has a (g, f) -factor if and only if, for all disjoint subsets S and T of $V(G)$, it follows*

$$\sum_{x \in S} f(x) \geq \sum_{x \in T} (g(x) - d_{G-S}(x)).$$

The above criterion can be deduced from Lovász's (g, f) -Factor Theorem. However, we present a shorter and direct structural proof by Heinrich *et al.* [239] and Addario-Berry *et al.* [3], independently. We restate it in an extensive form. This form is handy to use in the future applications (e.g., see Section 2.4).

Theorem 2.1.5 (Heinrich *et al.* [239]; Addario-Berry *et al.* [3]). *Let G be a graph and $g, f : V(G) \rightarrow \mathbb{Z}^+$ such that $0 \leq g(x) < f(x)$ for all $x \in V$. Then, precisely one of the following holds:*

- (i) G has a (g, f) -factor, or
- (ii) there exist subsets $S, T \subseteq V(G)$, $S \cap T = \emptyset$ such that

$$\sum_{x \in T} (g(x) - d_{G-S}(x)) > \sum_{x \in S} f(x). \quad (2.5)$$

Furthermore, if (ii) holds, then there exists a pair of sets S and T satisfying (2.5) such that for any $w \in T$, we can choose a $(0, f)$ -factor F , which attain the total deficiency of G

$$\text{def}(G) = \sum_{x \in V(F)} \max\{0, g(x) - d_F(x)\}$$

and satisfies:

- (a) $d_F(x) \leq g(x)$ for any $x \in T$;
- (b) $d_F(x) = f(x)$ for any $x \in S$;
- (c) $d_F(w) < g(w)$.

Proof. Assume that (i) holds, i.e., G has a (g, f) -factor F . For any pair of disjoint sets $S, T \subseteq V(G)$, consider the number of edges, $e_F(S, T)$, between S and T in F . If $x \in T$, then there are at least $d_F(x) - d_{G-S}(x) \geq g(x) - d_{G-S}(x)$ edges from x to S . Thus

$$e_F(S, T) \geq \sum_{x \in T} (g(x) - d_{G-S}(x)).$$

On the other hand, $e_F(S, T) \leq \sum_{x \in S} d_F(x) \leq \sum_{x \in S} f(x)$. That is, there does not exist a pair of S and T satisfying (2.5).

To prove the sufficiency, assume that G has no (g, f) -factors, and let F be a $(0, f)$ -factor which attains the total deficiency of G , i.e.,

$$\text{def}(F) = \text{def}(G) = \sum_{x \in V(F)} \max\{0, g(x) - d_F(x)\}.$$

Let $T_0 = \{x \mid d_F(x) < g(x)\}$. Clearly, T_0 is nonempty. A u -alternating path with respect to F is a sequence of vertices, $u_1, u_2, \dots, u_k (= u)$, such that $u_1 \in T_0$, and for every i , $u_{2i-1}u_{2i} \in E(G - F)$, $u_{2i}u_{2i+1} \in E(F)$. Let S

(resp. T) be the set of vertices u for which there exists an odd (resp. even) u -alternating path. Clearly, $T_0 \subseteq T$. By the minimality of deficiency of F , for any vertex $x \in T$, $d_F(x) \leq g(x)$, and for any vertex $x \in S$, $d_F(x) = f(x)$. Otherwise, let P be a x -alternating path, then the symmetric difference of F and P , $F \triangle P$, would be a $(0, f)$ -factor with smaller deficiency. Therefore, (a) and (b) hold. By the assumption of $g(x) < f(x)$, S and T are disjoint. From the definitions of S and T , if xy is an edge in F , then $x \in S$ implies $y \in T$, and if xy is an edge in $G - F$, then $x \in T$ implies $y \in S$. Thus, for any $x \in T$,

$$d_{G-S}(x) = d_F(x) - e_F(x, S) \leq g(x).$$

Therefore,

$$\begin{aligned} \sum_{x \in T} (g(x) - d_{G-S}(x)) &= \sum_{x \in T} (g(x) - d_F(x)) + \sum_{x \in T} e_F(x, S) \\ &> \sum_{x \in T} e_F(x, S) = \sum_{x \in S} f(x). \end{aligned}$$

By the definition of T_0 , $d_F(w) < g(w)$ or (c) holds, for any $w \in T_0$. For any vertex $w \in T - T_0$, let P be a w -alternating path. It is easy to check that the symmetric difference $F \triangle P$ defines a desired pair S and T and satisfies (a)–(c). $\square$

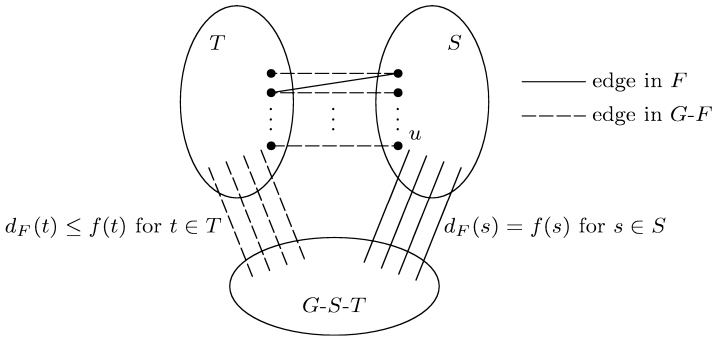


Fig. 2.2 Illustration of the proof of Theorem 2.1.5.

There is an even more simpler criterion for (g, f) -factor under the condition “ $g(x) < f(x)$ for all $x \in V(G)$ ”. Interesting enough, this time, the criterion only uses single subset S of $V(G)$ just like in Tutte’s 1-Factor Theorem, rather than examining a pair of disjoint sets. The proof of this theorem can be deduced from Theorem 2.1.5 easily.

Theorem 2.1.6 (Heinrich, Hell, Kirkpatrick and Liu [239]). *Let G be a graph and $g, f : V(G) \rightarrow \mathbb{Z}^+$. Assume that one of the following two conditions holds:*

- (a) $g(x) < f(x)$ for all $x \in V$;
- (b) G is bipartite.

Then G has a (g, f) -factor if and only if for any $S \subseteq V(G)$

$$\sum_{x \in S} f(x) \geq \sum_{y \notin S} \max\{0, g(y) - d_{G-S}(y)\}.$$

Another special case of (g, f) -factors is when $0 \leq g(x) \leq 1$, which is a generalization of 1-factors. It is interesting to notice that the following criterion is only involved with examination of one set too.

Theorem 2.1.7 (Las Vergnas [334]). *Let G be a graph and $g, f : V(G) \rightarrow \mathbb{Z}^+$ such that $0 \leq g(x) \leq 1$ and $g(x) \leq f(x)$ for all $x \in V(G)$. Then G has a (g, f) -factor if and only if for any $S \subseteq V(G)$*

$$\tilde{o}(G - S) \leq \sum_{x \in S} f(x),$$

where $\tilde{o}(G - S)$ denotes the number of components C of $G - S$ such that either (a) $C = \{z\}$ and $g(z) = 1$; or (b) $|C|$ is odd, $|C| \geq 3$ and $g(x) = f(x) = 1$ for all $x \in V(C)$.

For two constant integers a and b with $0 < a < b$, by setting $g(x) = a$ and $f(x) = b$ for all the vertices x , we can get the following $[a, b]$ -Factor Theorem as a corollary from (g, f) -Factor Theorem. Note that $q_G(S, T) = 0$ for all pairs (S, T) by noticing $a < b$, and so the criterion becomes simpler.

Theorem 2.1.8 ([a, b]-Factor Theorem with $a < b$). *Let G be a graph and a and b be integers such that $0 < a < b$. Then G has an $[a, b]$ -factor if and only if for all disjoint subsets S and T of $V(G)$,*

$$\delta_G(S, T) = b|S| + \sum_{x \in T} d_G(x) - a|T| - e_G(S, T) \geq 0. \quad (2.6)$$

Another version of Theorem 2.1.8 is observed by Heinrich *et al.* (1990), which uses only one subset S of G instead of two disjoint subsets S and T of $V(G)$.

Theorem 2.1.9 (Heinrich, Hell, Kirkpatrick and Liu [239]). *Let $b > a \geq 1$ be integers. Then a graph G has an $[a, b]$ -factor if and only if for all $S \subseteq V(G)$*

$$\sum_{i=0}^{a-1} (a-i)p_i(G-S) \leq b|S|, \quad (2.7)$$

where $p_i(G - S)$ is the number of vertices of degree i in subgraph $G - S$.

Remark 2.1.3. It is clear that (2.7) and the following are equivalent:

$$a|W| - \sum_{x \in W} d_{G-S}(x) \leq b|S|,$$

where $W = \{x \in V(G - S) \mid d_{G-S}(x) < a\}$.

To conclude this section, we present the following two interesting sufficient conditions for (g, f) -factors, by applying the above simple criterion. The first one is a ratio type condition, which was proven in [239] and [175], independently. The second one is due by Kano and Saito (1983).

Theorem 2.1.10 (Heinrich *et al.* [239]; Egawa and Kano [175]). *Let G be a graph and $g, f : V(G) \rightarrow \mathbb{Z}^+$. If for all distinct vertices x and y of G ,*

$$\frac{g(y)}{d_G(y)} \leq \frac{f(x)}{d_G(x)}.$$

Then G has a (g, f) -factor.

Proof. At first, note the fact

$$\sum_{x \notin S} (d_G(x) - d_{G-S}(x)) \leq \sum_{x \in S} d_G(x) \quad \text{for any set } S \subseteq V(G).$$

Set $W = \{x \notin S \mid g(x) \geq d_{G-S}(x)\}$. Hence, we have

$$\begin{aligned} & f(S) - \sum_{x \notin S} (g(x) - d_{G-S}(x)) \\ & \geq \frac{f(S)}{d_G(S)} \sum_{x \notin S} (d_G(x) - d_{G-S}(x)) - \sum_{x \notin S} (g(x) - d_{G-S}(x)) \\ & = \frac{1}{d_G(S)} \left(\sum_{x \in W} ((f(S)d_G(x) - d_G(S)g(x)) + (d_G(S) - f(S))d_{G-S}(x)) \right) \\ & \quad + \frac{1}{d_G(S)} \left(f(S) \sum_{x \notin S \cup W} ((d_G(x) - d_{G-S}(x)) + d_G(S)(d_{G-S}(x) - g(x))) \right) \\ & \geq 0. \end{aligned}$$

From Theorem 2.1.6, G has a (g, f) -factor. □

Theorem 2.1.11 (Kano and Saito [288]). *Let G be a connected graph and $g, f : V(G) \rightarrow \mathbb{Z}^+$ such that $g(x) < f(x)$ for all $x \in V(G)$. If there exists a real number $0 \leq \theta \leq 1$ such that*

$$g(x) \leq \theta d_G(x) \leq f(x) \quad \text{for all } x \in V(G), \tag{2.8}$$

then G has a (g, f) -factor.

Proof. Let S and T be two disjoint subset of $V(G)$. Then

$$\begin{aligned} \delta(S, T) &= \sum_{x \in S} f(x) + \sum_{x \in T} (d_G(x) - g(x)) - e_G(S, T) \\ &\geq \theta \sum_{x \in S} d_G(x) + \sum_{x \in T} (1 - \theta) d_G(x) - e_G(S, T) \\ &\geq \theta e_G(S, T) + (1 - \theta) e_G(S, T) - e_G(S, T) = 0. \end{aligned}$$

From Theorem 2.1.4, G has a (g, f) -factor. □

2.2 Factors in bipartite graphs

This section we discuss the factors in bipartite graphs or graphs acting like the bipartite graphs (i.e., the graphs sharing much factor properties as in bipartite graphs), for example the graphs with OCP.

A graph G is said to have the *odd cycle property* (OCP in short) if every pair of odd cycles in G either have a vertex in common or are joined by an edge (see Fig. 2.3). This property is first noticed by Fulkerson, Hoffman and McAndrew [219]. This family of graphs is a larger class of graphs than bipartite graphs but has many properties possessing only by bipartite graphs. We discuss this family briefly in Section 1.2, now we study the relations between this property and the existences of factors.

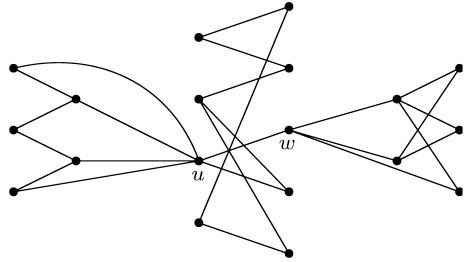


Fig. 2.3 A graph with OCP.

Let us start with 1-factors. The following theorem shows that Hall's condition is sufficient for the existence of 1-factors in graphs with OCP.

Theorem 2.2.1 (Berge (1973)). *Let G be a connected graph of even order with OCP. Then G has a 1-factor if and only if*

$$|N_G(S)| \geq |S| \quad \text{for all } S \subseteq V(G). \quad (2.9)$$

Similar to Theorem 1.2.2, we have a criterion for 1-factors in terms of isolated vertices.

Theorem 2.2.2. *Let G be a connected graph of even order with OCP. Then G has a 1-factor if and only if*

$$i(G - S) \leq |S| \quad \text{for all } S \subseteq V(G). \quad (2.10)$$

Ore (1957) obtained a criterion for f -factors in bipartite graphs, which is a simple formula without the tedious term $q_G(S, T)$. Fulkerson, Hoffman and McAndrew (1965) extended Ore's theorem to graphs with OCP. Comparing with Theorem 2.2.3, we see that the term $\delta(S, T)$ is almost identical to that in Theorem 2.2.4.

Theorem 2.2.3 (f -Factor Theorem in Bipartite Graphs, Ore [469]). *Let $G = (A, B)$ be a bipartite graph and $f : V(G) \rightarrow \mathbb{Z}^+$ with $f(A) = f(B)$. Then the following statements are equivalent:*

- (a) G has an f -factor;
- (b) for all sets $S \subset A$ and $T \subset B$, then

$$f(T) \leq f(S) + e(A - S, T);$$

- (c) for all $T \subset B$,

$$f(T) \leq \sum_{x \in A} \min\{f(x), e(T, x)\};$$

- (d) for all $S \subset A$ and $T \subset B$,

$$d_G(T) - f(T) + f(S) - e(S, T) \geq 0.$$

The condition (b) is also a sufficient and necessary condition for existence of f -factors in graphs with OCP.

Theorem 2.2.4 (f -Factor Theorem in OCP Graphs, Fulkerson, Hoffman and McAndrew [219]). *Let G be a graph with OCP and $f : V(G) \rightarrow \mathbb{Z}^+$. Then G has an f -factor if and only if $\sum_{x \in V(G)} f(x)$ is even and for all disjoint subsets S and T of $V(G)$,*

$$\delta(S, T) = f(S) + d_{G-S}(T) - f(T) \geq 0. \quad (2.11)$$

Matchings and factor problems of bipartite graphs have many applications in Management Science, Operation Researches, and Combinatorial Optimization (e.g., the job assignment problem). For this reason, we highlight the following theorem by Folkman and Fulkerson [212]. Note that some parts of this theorem are also proved by other authors such as Ore (1957), Gale (1957) and Ryser (1957).

Theorem 2.2.5 ((g, f)-Factor Theorem in Bipartite Graphs, Folkman and Fulkerson [212]). *Let $G = (A, B)$ be a bipartite multigraph, and $g, f : A \cup B \rightarrow \mathbb{Z}^+$ such that $g(v) \leq f(v)$ for all $v \in A \cup B$. Then G has a (g, f) -factor if and only if for all $S \subset A$ and $T \subset B$,*

- (a) $g(T) \leq f(S) + e(A - S, T)$, and
- (b) $g(S) \leq f(T) + e(B - T, S)$.

Note that the condition (a) in the above theorem can be replaced by the following two conditions:

$$g(T) \leq \sum_{x \in A} \min\{f(x), e(T, x)\} \quad \text{for all } T \subset B;$$

and

$$d_G(T) - g(T) + f(S) - e(S, T) \geq 0 \quad \text{for all } S \subset A \text{ and } T \subset B.$$

With minor modifications to Theorem 2.2.5, Chen and Wang (1993) obtained a characterization of (g, f) -factors for graphs with OCP. Note that the assumption “ $g(w) < f(w)$ for some vertex w ” in Theorem 2.2.6 is a reasonable one, since otherwise a (g, f) -factor becomes an f -factor.

Theorem 2.2.6 (Chen and Wang [141]). *Let G be a connected graph with OCP. Let $g, f : V(G) \rightarrow \mathbb{Z}^+$ such that $g(x) \leq f(x)$ for all $x \in V(G)$ and $g(w) < f(w)$ for some vertex w . Then G has a (g, f) -factor if and only if for all disjoint subsets S and T of $V(G)$ with $S \cup T \neq \emptyset$, we have*

$$\delta(S, T) = \sum_{x \in S} f(x) + \sum_{x \in T} (d_G(x) - g(x)) - e_G(S, T) \geq \varepsilon_0, \quad (2.12)$$

where $\varepsilon_0 = 1$ if G has an odd cycle C with $g(x) = f(x)$ for all vertices $x \in V(C)$; otherwise $\varepsilon_0 = 0$.

The following theorem gives a criterion for $[a, b]$ -factors in bipartite graphs, which is obtained by Enomoto, Ota and Kano (1988).

Theorem 2.2.7 (Enomoto, Ota and Kano [190]). *Let $a \geq 2$ and $b \geq 2$ be integers. Let $G = (A, B)$ be a bipartite graph. Then G has a spanning subgraph F such that $d_G(x) = a$ for all $x \in A$ and $d_G(y) \leq b$ for all $y \in B$ if and only if for all sets $S \subseteq A$ and $T \subseteq B$,*

$$\delta(S, T) = b|T| + \sum_{x \in S} d_{G-T}(x) - a|S| \geq 0. \quad (2.13)$$

The following is a related result to the above theorem.

Theorem 2.2.8. *A bipartite graph $G = (A, B)$ has an f -factor with $f(x) = 1$ for all $x \in A$ and $f(y) \leq k$ for all $y \in B$ if and only if*

$$|N(S)| \geq |S|/k \quad \text{for all } S \subseteq A.$$

Under odd-cycle property, the graphs with k -factor have an interesting feature, pan-factorial property. This result is observed by Anstee (1992) and can be proved easily from Theorem 8.2.2.

Theorem 2.2.9 (Anstee [60]). *Let G be a graph with odd-cycle property. Suppose G has a k -factor. Then G has an r -factor for*

- (a) *any even integer $r \leq k$, or*
- (b) *any odd integer r satisfying $1 \leq r < k$ when $|V(G)|$ is even.*

2.3 Factors with special properties

The graphs with factors possess many interesting properties, such as pan-factorial property (or called interpolation property), containing a given substructure, implication from smaller factor to a larger factor, and so on and so forth. In this section, we investigate these properties respectively.

Before we enter the investigation, let us recall one of the most classic results due to Petersen (1891). If the edge set $E(G)$ of a graph G is decomposed into edge disjoint k -regular factors F_i ($1 \leq i \leq m$), i.e.,

$$E(G) = F_1 \cup F_2 \cup \cdots \cup F_m, \quad F_i \cap F_j = \emptyset \quad (i \neq j),$$

then we call $F_1, F_2, \dots, F_m$ a k -factorization of G .

Theorem 2.3.1 (2-Factorization Theorem, Petersen [473]). *For any integer $r \geq 1$, every $2r$ -regular graph has a 2-factorization. In particular, for every integer k , $1 \leq k \leq r$, every $2r$ -regular graph has a $2k$ -regular factor.*

Note that the above theorem also holds for the graphs with loops and multiple edges.

Now we turn to the investigation of continuity property for k -factors. The *continuity property* is that if a graph has an r -factor and an s -factor, where $r \leq s$, then it must have a k -factor for all k , $r \leq k \leq s$. Sometimes, we also call it *pan-factorial property* or *interpolation property*. For instance, Theorem 2.2.9 in the last section is such a theorem.

Theorem 2.3.2 (Katerinis [294]). *Let a, b, k be odd integers such that $1 \leq a \leq k \leq b$. If a graph G has both a -factors and b -factors, then G has a k -factor.*

Proof. Suppose, to the contrary, that G does not have k -factors. Then by k -Factor Theorem, there exists a pair of disjoint subsets, S and T , of $V(G)$ such that

$$\sum_{x \in T} (k - d_{G-S}(x)) + q_G(S, T; k) > k|S|. \quad (2.14)$$

But G has an a -factor and a b -factor, so by the same theorem, we have

$$\sum_{x \in T} (a - d_{G-S}(x)) + q_G(S, T; a) \leq a|S| \quad (2.15)$$

and

$$\sum_{x \in T} (b - d_{G-S}(x)) + q_G(S, T; b) \leq b|S|. \quad (2.16)$$

Because if C is a component of $(G - S) - T$, the quantities $e(C, T) + k|V(C)|$, $e(C, T) + a|V(C)|$ and $e(C, T) + b|V(C)|$ have the same parity, we have $q_G(S, T; k) = q_G(S, T; a) = q_G(S, T; b)$. So from (2.14) and (2.15) we have that $|T| > |S|$, and (2.14) and (2.16) imply that $|S| > |T|$, which is a contradiction. $\square$

Corollary 2.3.1. *If an r -regular graph G has a 1-factor, then for every integer $k \in \{1, \dots, r\}$, G has a k -factor.*

Proof. Let G is an r -regular graph with a 1-factor F . The $G - F$ is an $(r - 1)$ -factor. From Theorem 2.3.2, G has all the k -factors for $1 \leq k \leq r - 1$. $\square$

The next result is about the continuity property of $[a, b]$ -factors. It can also be considered as a generalization of Theorem 3.4.6 by Thomassen (1981).

Theorem 2.3.3 (Kano and Saito [288]). *Let l, k, r, t be integers such that $0 \leq l < k < r$ and $t \geq 1$. Then for any given $[l, l + t]$ -factor H of an $[r, r + t]$ -graph G , G has a $[k, k + t]$ -factor that contains H as a subgraph.*

Proof. Let G be an $[r, r + t]$ -graph, H be an $[l, l + t]$ -factor of G , and $K = G - E(H)$. Define two functions $g, f : V(K) \rightarrow \mathbb{Z}^+$ by $g(x) = k - d_H(x)$ and $f(x) = k + t - d_H(x)$ for all $x \in V(G)$. If K has a (g, f) -factor F , then $F \cup H$ is the desired $[k, k + t]$ -factor. Let

$$\theta = \frac{k - l}{r - l}.$$

We shall show K , g and f satisfy condition (2.8) in Theorem 2.1.11.

Clearly, $0 \leq \theta \leq 1$ and $g(x) < f(x)$. Since

$$k = \theta r + (1 - \theta)l \leq \theta d_G(x) + (1 - \theta)d_H(x)$$

and

$$d_K(x) = d_G(x) - d_H(x),$$

we have

$$\begin{aligned} g(x) &= k - d_H(x) \leq \theta d_G(x) + (1 - \theta)d_H(x) - d_H(x) \\ &= \theta(d_G(x) - d_H(x)) = \theta d_K(x). \end{aligned}$$

Similarly, since

$$k + t = \theta(r + t) + (1 - \theta)(l + t) \geq \theta d_G(x) + (1 - \theta)d_H(x),$$

we have

$$\begin{aligned} f(x) &= k + t - d_H(x) \\ &\geq \theta d_G(x) + (1 - \theta)d_H(x) - d_H(x) \\ &= \theta(d_G(x) - d_H(x)) = \theta d_K(x). \end{aligned}$$

Hence $g(x) \leq \theta d_K(x) \leq f(x)$ or (2.8) holds. The proof is complete. $\square$

We continue to study the pan-factorial (or continuity) property. Given two integer functions g and f on $V(G)$ with $g(x) \leq f(x)$, for any integer function h with $g(x) \leq h(x) \leq f(x)$ and $\sum_{x \in V} h(x) \equiv 0 \pmod{2}$, we call an h -factor of G a *middle factor*. Niessen (1998) provided a necessary and sufficient condition for the existence of middle factors.

Theorem 2.3.4 (Niessen [448]). *Let G be a graph and $g, f : V(G) \rightarrow \mathbb{Z}^+$ with $g(x) \leq f(x)$ for all $x \in V(G)$ and $g(w) < f(w)$ for at least one vertex w . Then G has a middle h -factor if and only if for any two disjoint subsets S and T of $V(G)$,*

$$\delta_G^*(S, T) = \sum_{x \in S} g(x) - \sum_{x \in T} f(x) + \sum_{x \in T} d_{G-S}(x) - q^*(S, T) \geq -1, \quad (2.17)$$

where $q^*(S, T)$ is the number of components C of $G - (S \cup T)$ such that, either $g(u) < f(u)$ for some vertex $u \in V(C)$; or $g(x) = f(x)$ for all $x \in V(C)$ and

$$\sum_{x \in V(C)} f(x) + e_G(C, T) \equiv 1 \pmod{2}.$$

Proof. We first verify that (2.17) ensures that G has all middle factors. Let $h : V(G) \rightarrow \mathbb{Z}^+$ satisfy $g(x) \leq h(x) \leq f(x)$ for every $x \in V(G)$ and $\sum_{x \in V(G)} h(x) \equiv 0 \pmod{2}$. Then for two disjoint subsets S, T of $V(G)$,

$$\begin{aligned} \delta(S, T; h) &= \sum_{x \in S} h(x) - \sum_{x \in T} h(x) + \sum_{x \in T} d_{G-S}(x) - q(S, T; h) \\ &\geq \sum_{x \in S} g(x) - \sum_{x \in T} f(x) + \sum_{x \in T} d_{G-S}(x) - q^*(S, T; g, f) \\ &\geq -1. \end{aligned}$$

So, by f -Factor Theorem, G has an h -factor.

Next, we show that (2.17) is satisfied if G has all middle factors. Let $S, T \subseteq V(G)$ be two disjoint subsets. Define $h : V(G) \rightarrow \mathbb{Z}^+$ as follow. For all $x \in S$, let $h(x) = g(x)$; and for all $x \in T$, let $h(x) = f(x)$. Now let C be a component of $G - (S \cup T)$. If $f(x) = g(x)$ for all $x \in V(C)$, then let $h(x) = f(x)$ for all $x \in V(C)$. Otherwise there exists a vertex $u \in V(C)$ such that $g(u) < f(u)$. Then we let $h(x) = f(x)$ for all $x \in V(C) - \{u\}$ and choose $h(u)$ to be $f(u)$ or $f(u) - 1$ such that $e_G(C, T) + \sum_{x \in V(C)} h(x) \equiv 1 \pmod{2}$. Therefore, $q^*(S, T; g, f) = q(S, T; h)$, and so

$$\sum_{x \in S} g(x) - \sum_{x \in T} f(x) + \sum_{x \in T} d_{G-S}(x) - q^*(S, T; g, f) = \delta(S, T; h).$$

Hence S and T satisfy (2.17) by f -Factor Theorem, if $\sum_{x \in V(G)} h(x)$ is even, since G has an h -factor. Finally, if $\sum_{x \in V(G)} h(x)$ is odd, we have $f \neq g$, since G has a (g, f) -factor. So there exists a vertex $u \in V(G)$ such that $g(u) < h(u)$ or $h(u) < f(u)$. Now we define $h^* : V(G) \rightarrow \mathbb{Z}^+$ by $h^*(x) = h(x)$ for all $x \in V(G) - \{u\}$ and

$$h^*(u) = \begin{cases} h(u) - 1 & \text{if } h(u) = f(u); \\ h(u) + 1 & \text{otherwise.} \end{cases}$$

Therefore, $\sum_{x \in V(G)} h^*(x)$ is even and $g(x) \leq h^*(x) \leq f(x)$ holds for all $x \in V(G)$. So, G has an h^* -factor and thus $\delta(S, T; h^*) \geq 0$. Moreover, it holds that

$$\delta(S, T; h) - \delta(S, T; h^*) = \begin{cases} h(u) - h^*(u) & \text{if } u \in S; \\ h^*(u) - h(u) & \text{if } u \in T; \\ q(S, T; h^*) - q(S, T; h) & \text{otherwise,} \end{cases}$$

and so we have $\delta(S, T; h) - \delta(S, T; h^*) \geq -1$. Thus

$$\begin{aligned} \sum_{x \in S} g(x) - \sum_{x \in T} f(x) + \sum_{x \in T} d_{G-S}(x) - q^*(S, T; g, f) &= \delta(S, T; h) \\ &\geq \delta(S, T; h^*) - 1 \geq -1. \end{aligned}$$

Since $f \neq g$, this shows that (2.17) is also satisfied, if $\sum_{x \in V(G)} h(x)$ is odd.

We complete the proof of the theorem. $\square$

Note that the condition “ $g(w) < f(w)$ for at least one vertex w ” is not a real restriction. Since, otherwise, $g(x) = f(x)$ for all $x \in V(G)$, then the middle factor is nothing but an f -factor.

Niessen [448] also considered the parity (g, f) -factors as the middle factors and obtained the following criterion. The proof of Theorem 2.3.5 is similar to that of Theorem 2.3.4 but simpler. Let us recall that a spanning subgraph

H is called a *parity* (g, f) -factor if $g(x) \leq d_H(x) \leq f(x)$ and $d_H(x) \equiv f(x) \pmod{2}$ for all $x \in V(H)$, where we assume that $g(x) \equiv f(x) \pmod{2}$ for all $x \in V(G)$.

Theorem 2.3.5 (Niessen [448]). *A graph G has a middle (g, f) -parity-factor if and only if for any two disjoint subsets S and T of $V(G)$,*

$$\delta_G(S, T) = \sum_{x \in S} g(x) - \sum_{x \in T} f(x) + \sum_{x \in T} d_{G-S}(x) - q(S, T; f) \geq 0, \quad (2.18)$$

where $q(S, T; f)$ is the same as in f -Factor Theorem.

Now we turn to the continuity property of $[a, b]$ -factors.

Theorem 2.3.6 (Kano [277]). *Let a, b, c, d, m, n be integers such that $0 \leq a < m < c$ and $a < b, c < d$ and $m + 1 \leq n$. Suppose that a graph G has both an $[a, b]$ -factor and a $[c, d]$ -factor. If*

$$\frac{n - b}{m - a} \geq \frac{n - d}{m - c},$$

then G has an $[m, n]$ -factor.

Proof. Suppose, to the contrary, that G has no $[m, n]$ -factors. By Theorem 2.1.4, there exist two disjoint subsets S, T of $V(G)$ such that

$$\delta(S, T) = n|S| + \sum_{x \in T} d_{G-S}(x) - m|T| < 0.$$

Since G has an $[a, b]$ -factor and a $[c, d]$ -factor, by Theorem 2.1.8, then

$$\delta_1(S, T) = b|S| + \sum_{x \in T} d_{G-S}(x) - a|T| \geq 0,$$

$$\delta_2(S, T) = d|S| + \sum_{x \in T} d_{G-S}(x) - c|T| \geq 0.$$

Hence we have

$$0 > \delta(S, T) - \delta_1(S, T) \geq (n - b)|S| - (m - a)|T|, \quad (2.19)$$

$$0 > \delta(S, T) - \delta_2(S, T) \geq (n - d)|S| - (m - c)|T|. \quad (2.20)$$

If $S = \emptyset$, then $0 > -(m - c)|T|$ by (2.20), a contradiction to $m < c$. Thus $S \neq \emptyset$. By (2.19) and (2.20), we have

$$\frac{n - b}{m - a} < \frac{|T|}{|S|} \quad \text{and} \quad \frac{|T|}{|S|} < \frac{n - d}{m - c},$$

or

$$\frac{n-b}{m-a} < \frac{n-d}{m-c},$$

a contradiction to the assumption. Therefore, G has an $[m, n]$ -factor. $\square$

The above theorem can be extended further to include or exclude a given edge, which was obtained by Kano and Yu (2005).

Theorem 2.3.7 (Kano and Yu [290]). *Let G be a connected graph of even order, e be any edge of G , and a, b, k be odd integers such that $1 \leq a < k < b$. If G has both an a -factor and a b -factor containing e , then G has a k -factor containing e . Similarly, if G has both an a -factor and a b -factor avoiding e , then G has a k -factor avoiding e .*

Proof. Let e be an edge of G . Assume that G has both an a -factor and a b -factor avoiding e . By applying Theorem 2.3.2 to $G - e$, we see that $G - e$ has a k -factor, which implies that G has a k -factor avoiding e .

We next show that if G has both an a -factor and a b -factor containing e , then G has a k -factor containing e .

Define a new graph G^* by inserting a new vertex w on the edge e , and define an integer-value function $f_c : V(G^*) \rightarrow \{c, 2\}$ such that

$$f_c(x) = \begin{cases} c & \text{if } x \in V(G); \\ 2 & \text{if } x = w. \end{cases}$$

Then G has a c -factor containing e if and only if G^* has an f_c -factor. It is obvious that $\sum_{x \in V(G^*)} f_c(x) = c|V(G)| + 2 \equiv 0 \pmod{2}$ since G is of even order.

Assume that G^* has no f_k -factors. Then, by f -Factor Theorem, there exist two disjoint subsets $S, T \subseteq V(G^*)$ such that

$$\delta(S, T; f_k) = \sum_{x \in S} f_k(x) + \sum_{x \in T} (d_{G^*-S}(x) - f_k(x)) - q(S, T; f_k) \leq -2. \quad (2.21)$$

On the other hand, since G^* has both an f_a -factor and an f_b -factor, we have

$$\delta(S, T; f_a) = \sum_{x \in S} f_a(x) + \sum_{x \in T} (d_{G^*-S}(x) - f_a(x)) - q(S, T; f_a) \geq 0, \quad (2.22)$$

$$\delta(S, T; f_b) = \sum_{x \in S} f_b(x) + \sum_{x \in T} (d_{G^*-S}(x) - f_b(x)) - q(S, T; f_b) \geq 0. \quad (2.23)$$

Now depending on the location of w , we consider three cases:

Case 1. $w \notin S \cup T$.

(2.21), (2.22) and (2.23) can be rewritten as

$$k|S| + \sum_{x \in T} d_{G^*-S}(x) - k|T| - q(S, T; f_k) \leq -2, \quad (2.24)$$

$$a|S| + \sum_{x \in T} d_{G^*-S}(x) - a|T| - q(S, T; f_a) \geq 0, \quad (2.25)$$

$$b|S| + \sum_{x \in T} d_{G^*-S}(x) - b|T| - q(S, T; f_b) \geq 0. \quad (2.26)$$

Subtracting (2.25) from (2.24), we have

$$(k - a)(|S| - |T|) + q(S, T; f_a) - q(S, T; f_k) \leq -2. \quad (2.27)$$

Similarly, from (2.26) and (2.24), we have

$$(k - b)(|S| - |T|) + q(S, T; f_b) - q(S, T; f_k) \leq -2. \quad (2.28)$$

Recall that $q(S, T; f_c)$ is the number of f_c -odd components C of $G^* - (S \cup T)$, which satisfies $\sum_{x \in V(C)} f_c(x) + e_{G^*}(C, T) \equiv 1 \pmod{2}$. Since all a, b and k are odd integers, it follows that if $w \notin V(C)$, then

$$\begin{aligned} \sum_{x \in V(C)} f_a(x) + e_{G^*}(C, T) &= a|C| + e_{G^*}(C, T) \\ &\equiv b|C| + e_{G^*}(C, T) = \sum_{x \in V(C)} f_b(x) + e_{G^*}(C, T) \pmod{2} \\ &\equiv k|C| + e_{G^*}(C, T) = \sum_{x \in V(C)} f_k(x) + e_{G^*}(C, T) \pmod{2}. \end{aligned}$$

Therefore we obtain

$$q(S, T; f_k) - q(S, T; f_a) \leq 1 \quad \text{and} \quad q(S, T; f_k) - q(S, T; f_b) \leq 1.$$

If $|S| \geq |T|$, then (2.27) implies

$$-1 \leq (k - a)(|S| - |T|) + q(S, T; f_a) - q(S, T; f_k) \leq -2,$$

a contradiction. If $|S| < |T|$, then (2.28) implies

$$-1 \leq (k - b)(|S| - |T|) + q(S, T; f_b) - q(S, T; f_k) \leq -2,$$

a contradiction again.

Case 2. $w \in S$.

In this case, (2.21), (2.22) and (2.23) become

$$\begin{aligned}
2 + k(|S| - 1) + \sum_{x \in T} d_{G^* - S}(x) - k|T| - q(S, T; f_k) &\leq -2, \\
2 + a(|S| - 1) + \sum_{x \in T} d_{G^* - S}(x) - a|T| - q(S, T; f_a) &\geq 0, \\
2 + b(|S| - 1) + \sum_{x \in T} d_{G^* - S}(x) - b|T| - q(S, T; f_b) &\geq 0.
\end{aligned}$$

It is clear that $q(S, T; f_k) = q(S, T; f_a) = q(S, T; f_b)$. If $|S| \geq |T| + 1$, we have $0 \leq (k - a)(|S| - 1 - |T|) \leq -2$, a contradiction; if $|S| < |T| + 1$, then $0 \leq (k - b)(|S| - 1 - |T|) \leq -2$, a contradiction as well.

Case 3. $w \in T$.

In this case, (2.21), (2.22) and (2.23) become

$$\begin{aligned}
k|S| + \sum_{x \in T} d_{G^* - S}(x) - 2 - k(|T| - 1) - q(S, T; f_k) &\leq -2, \\
a|S| + \sum_{x \in T} d_{G^* - S}(x) - 2 - a(|T| - 1) - q(S, T; f_a) &\geq 0, \\
b|S| + \sum_{x \in T} d_{G^* - S}(x) - 2 - b(|T| - 1) - q(S, T; f_b) &\geq 0.
\end{aligned}$$

Discussing similarly as in Case 2, we yield contradictions. Consequently the theorem is proved. $\square$

Next we turn to the property that the existence of a factor relates to the existence in its induced subgraphs.

Fraisse, Hell and Kirkpatrick (1986) showed that the existence of 2-factors in two subgraphs can ensure the existence of 2-factors in the whole graph.

Theorem 2.3.8 (Fraisse, Hell and Kirkpatrick [213]). *Let uv be an edge of a graph G . If both $G - u$ and $G - v$ have a 2-factor, then G also has a 2-factor.*

Katerinis (1983) considered the existence of even-factors in terms of vertex-deleted subgraphs.

Theorem 2.3.9 (Katerinis [292]). *Let G be a graph and r be an integer. Suppose $G - x$ has a $2r$ -factor for each vertex x of G . Then G also has a $2r$ -factor.*

Note that Theorem 2.3.9 only dealt with $2r$ -factors. For general k -factors, we need more subgraphs than those in the previous theorem to ensure the existence of k -factors.

Theorem 2.3.10 (Egawa, Enomoto and Saito [172]). *Let G be a graph and k be an integer. If $G - \{x, y\}$ has a k -factor for every edge $xy \in E(G)$, then G itself has a k -factor.*

In fact, Egawa, Enomoto and Saito (1988) proved a more general result than Theorem 2.3.10, shown in the next theorem.

Theorem 2.3.11 (Egawa, Enomoto and Saito [172]). *Let G be a connected graph of order n . Let k and p be positive integers. Suppose $0 < p < n$ and kn is even. If $G - V(H)$ has a k -factor for each connected induced subgraph of order p , then G also has a k -factor.*

In Theorem 2.3.11, the number of subgraphs required to check is $|E(G)|$. However, if a graph G has a 1-factor, then the number of subgraphs required to check is reduced to $|V(G)|/2$, which is proved by Saito (1991). To prove this result, we need a technical lemma from [172].

Lemma 2.3.1. *Let G be a graph and S, T be two disjoint subsets of $V(G)$. Then we have the following statements for $\delta_G(S, T)$ defined for k -factors,*

- (a) *if $x \in S$, then $\delta_{G-x}(S - x, T) = \delta_G(S, T) - k$;*
- (b) *if $x \in T$, then $\delta_{G-x}(S, T - x) \leq \delta_G(S, T) + k$;*
- (c) *if $X \subset V(G)$ for some component C of $G - (S \cup T)$, then $\delta_{G-X}(S, T) \leq \delta_G(S, T) + 1$.*

Theorem 2.3.12 (Saito [511]). *Let G be a graph of order at least four and k a positive integer. If G has a 1-factor F and for each $xy \in E(F)$, $G - \{x, y\}$ has a k -factor, then G itself has a k -factor.*

Proof. If G is disconnected, the theorem holds clearly. So we may assume that G is connected. Assume G has no k -factors. Then, by k -Factor Theorem (i.e., Theorem 2.1.1), we have

$$\delta_G(S, T) < 0 \quad \text{for some } S, T \subset V(G), S \cap T = \emptyset. \quad (2.29)$$

We may take a pair S, T satisfying (2.29) such that T is minimal. Let $U = V(G) - (S \cup T)$. Note $q_G(S, T) \leq |U|$. Since G has a 1-factor F , $|V(G)|$ is even. Hence $\delta_G(S, T) \leq -2$.

For $X \subset V(G)$, we use an abbreviation $\delta(X)$ to denote $\delta_{G-X}(S-X, T-X)$. According to this abbreviation, $\delta(X) < 0$ implies that $G-X$ has no k -factors. Furthermore, $\delta_G(S, T) = \delta(\emptyset)$.

Claim. If $x \in V(G)$ and $|N_G(x)| \geq 2$, then $d_G(x) \geq k + 1$.

Since $|V(G)| \geq 4$ and $G - V(e)$ has a k -factor for any $e \in F$, $d_G(x) \geq k$ for all $x \in V(G)$. Assume for some $x \in V(G)$, $|N_G(x)| \geq 2$ and $d_G(x) = k$. Let $y \in V(G)$ such that $xy \in F$ and take $z \in N_G(x) - \{y\}$. Then for some $u \in V(G)$, we have $zu \in F$. Since $d_{G-\{z, u\}}(x) \leq k - 1$, $G - \{z, u\}$ has no k -factors, a contradiction.

Next, we show that $S = \emptyset$. Assume $S \neq \emptyset$, say $x \in S$. Then for some $y \in V(G)$, $xy \in F$. If $y \in S$, then, by Lemma 2.3.1, we have

$$\delta(\{x, y\}) = \delta(\{x\}) - k = \delta(\emptyset) - 2k < 0.$$

Similarly, if $y \in T$, then

$$\delta(\{x, y\}) \leq \delta(\{x\}) + k = \delta(\emptyset) - k + k = \delta(\emptyset) < 0,$$

and if $y \in U$,

$$\delta(\{x, y\}) \leq \delta(\{x\}) + 1 \leq \delta(\emptyset) - k + 1 < 0.$$

Therefore, $\delta(\{x, y\}) < 0$ wherever y lies. This contradicts the assumption. Hence $S = \emptyset$.

Since $S = \emptyset$, $\delta(\emptyset) = \sum_{t \in T} (d_G(t) - k) - q_G(\emptyset, T)$. Suppose $xy \in F$ and $x, y \in U$. Then $\{x, y\} \subset C$ for some component C of $G - T$. However, this implies $\delta(\{x, y\}) \leq \delta(\emptyset) + 1 \leq -1$ by Lemma 2.3.1, another contradiction. Therefore, if $xy \in F$ and $x \in U$, then $y \in T$. Since F is a 1-factor, this implies $|U| \leq |T|$.

If $d_G(t) \geq k + 1$ for all $t \in T$, then

$$-2 \geq \delta(\emptyset) = \sum_{t \in T} (d_G(t) - k) - q_G(\emptyset, T) \geq |T| - |U|,$$

a contradiction. Hence $d_G(t) = k$ for some $t \in T$. By the claim, $|N_G(t)| = 1$. Now consider $\delta_G(\emptyset, T - \{t\})$,

$$\begin{aligned} \delta_G(\emptyset, T - \{t\}) &= \sum_{x \in T - \{t\}} d_G(x) - k|T - \{t\}| - q_G(\emptyset, T - \{t\}) \\ &= \sum_{x \in T} d_G(x) - d_G(t) - k|T| + k - q_G(\emptyset, T - \{t\}) \\ &= \delta_G(\emptyset, T) + q_G(\emptyset, T) - q_G(\emptyset, T - \{t\}). \end{aligned}$$

Since $|N_G(t)| = 1$, $q_G(\emptyset, T - \{t\}) \geq q_G(\emptyset, T) - 1$. Therefore, $\delta_G(\emptyset, T - \{t\}) \leq \delta_G(\emptyset, T) + 1 \leq -1$, a contradiction to the minimality of T . The last contradiction concludes the proof of the theorem. $\square$

Enomoto and Tokuda [192] extended the above result further to show that the existence of f -factor of G relies on the existence of f -factors of all subgraphs $G - V(C)$, where C is any component of $\{K_i \mid i \geq 1\}$ -factor in G .

The above theorems can be generalized to f -factors and (g, f) -factors as well. Here we present two such results.

For an integer-valued function f defined on $V(G)$ and a subgraph H , we use $f|_H$ to denote the restricted function of f on $V(H)$.

Theorem 2.3.13 (Fraisse, Hell and Kirkpatrick [213]). *Let G be a graph and R an induced subgraph of G . Let f be integer-valued function on $V(G)$ such that for some vertex $w \in V(R)$, $f(w)$ is even and $f(w) \leq$*

$d_R(w) + e_G(w, R - w)$. If for each $r \in V(R)$, $G - r$ has an $f|_{G-r}$ -factor, then G also has an f -factor.

Theorem 2.3.14 (Egawa, Enomoto and Saito [172]). *Let G be a graph of order n , g and f be integer-valued functions on $V(G)$, and p be an integer such that $0 < p < n$. Assume $0 \leq g(x) \leq f(x) \leq d_G(x)$ for all $x \in V(G)$. Suppose for each p -vertex induced subgraph H of G , H has a $(g|_H, f|_H)$ -factor. Then G has a (g, f) -factor unless $g(x) = f(x)$ for all $x \in V(G)$ and $\sum_{x \in V(G)} f(x)$ is odd.*

Next we consider the existences of 1-factors and k -factors with inclusion and exclusion property.

Theorem 2.3.15 (Kano and Yu [290]). *Let G be a connected r -regular graph of even order ($r \geq 2$). If for every edge e , G has a 1-factor containing e , then G has a k -factor containing e and another k -factor avoiding e for every $k \in \{1, 2, \dots, r-1\}$.*

Proof. For any given edge e , there exists a 1-factor F containing e . We choose another edge e' incident to the given edge e , then there exists a 1-factor F' containing e' and thus F' is a 1-factor avoiding e . Next, we construct the desired k -factors from F and F' .

Case 1. Assume that $r = 2m + 1$ is odd. From Petersen's Theorem, the $2m$ -regular graph $G - E(F)$ can be decomposed into disjoint 2-factors $T_1, T_2, \dots, T_m$, and $G - E(F')$ can be decomposed into disjoint 2-factors $T'_1, T'_2, \dots, T'_m$. Then $F \cup T_1 \cup \dots \cup T_j$ is a $(2j + 1)$ -factor containing e ($1 \leq j \leq m$). Without loss of generality, assume that $e \in T'_1$. Then $T'_1 \cup \dots \cup T'_j$ is a $2j$ -factor containing e ($1 \leq j \leq m$). That is, there exists a k -factor containing e for every $k \in \{1, 2, \dots, r-1\}$.

Similarly, we can obtain a k -factor avoiding e for every $k \in \{1, 2, \dots, r-1\}$.

Case 2. Assume that $r = 2m$ is even. From Petersen's Theorem, then G has all even factors containing e or avoiding e , respectively.

Since G has a 1-factor avoiding e , the graph $G - e$ has a 1-factor. In addition, $G - E(F)$ is a $(r-1)$ -factor of G avoiding e , and so $G - e$ has a $(r-1)$ -factor. Applying Theorem 2.3.2, we deduce that $G - e$ has all regular odd-factors between 1 and $r-1$, which are regular odd-factors of G avoiding e .

If H is a $(2j+1)$ -factor of G avoiding e ($1 \leq j \leq m-1$), then $G - E(H)$ is a $(2m - (2j+1))$ -factor containing e , which are all regular odd-factors of G , between 1 and $r-1$, containing e . $\square$

Now we discuss the problem of factors containing a given subgraph. We have seen such a result earlier (i.e., Theorem 2.3.3), in which there exists an $[a, b]$ -factor containing a given (smaller) $[c, d]$ -factor. Now we present one more such a result, which is more general than Theorem 2.3.3.

Theorem 2.3.16 (Kano and Saito [288]). *Let l, k, r, s, t, u be integers such that $0 \leq l \leq k \leq r$, $l + u \leq k + t \leq r + s$, $l \neq r$, $l + u \neq r + s$, $0 \leq s$, $0 \leq u$ and $1 \leq t$. Let G be an $[r, r + s]$ -graph and H be an $[l, l + u]$ -factor of G . If*

$$(l - k)s + (k - r)u + (r - l)t \geq 0, \quad (2.30)$$

then G has a $[k, k + t]$ -factor that contains the given factor H as a subgraph.

2.4 L -factor

It is well-known that finding a (g, f) -factor is a P-problem from algorithmic complexity point of view. However, the further generalization of (g, f) -factors may become an NP-complete problem. In this section, we discuss one of such generalizations, *list-factor*. This new factor has a close relation with edge-weight coloring problem (see, for example, [2], [4]). Moreover, itself constitutes an interesting research topic while it has many applications as well.

A (g, f) -factor is a subgraph with degrees to belong to prescribed *intervals* of integers. If we replace the intervals by *sets* of integers. We arrive a new generalization of f -factors and (g, f) -factors.

Let $L : V(G) \rightarrow 2^{\mathbb{N}}$ be a set-valued function, i.e., $L(v)$ is a set of integers for each vertex $v \in V(G)$. A *list factor*, or L -factor in short, of a graph G is a spanning subgraph H such that $d_H(v) \in L(v)$ for all $v \in V(G)$. We adopt the term “list-factor” as a set of integers can be viewed as a list and also its similar appearance with the well-known “list-coloring”. Among other terms used for this concept, sometimes it is called a *general factor* or H -factor. Without loss of generality, we assume that $L(v) \subseteq \{0, 1, \dots, d(v)\}$ for every $v \in V$.

If $F(v) \subseteq \mathbb{N}$ is a set of forbidden degrees for every $v \in V$, then a subgraph $G' = (V, E')$ of G is called F -avoiding if $d_{G'}(v) \notin F(v)$ for all $v \in V(G)$. Let $F(v) = \{0, 1, \dots, d_G(v)\} - L(v)$ for all $v \in V(G)$. Clearly, G has an L -factor if and only if G has an F -avoiding subgraph. Note that in several literatures (e.g., [403]) F -avoiding subgraphs are also referred as “anti-factors”.

In the case $|L(v)| = 1$ for all $v \in V$, f -Factor Theorem gives a necessary and sufficient condition for the existence of such a factor. However, there is no a necessary and sufficient condition known for the existence of an L -factor when we allow $|L(v)| \geq 2$, even when $|L(v)| = 2$ for all $v \in V$, since the decision problem of determining whether a graph has an L -factor is known to be NP-complete. In fact, Lovász [404] showed that the 3-edge-colorability problem is reducible to L -factor problem with $L(v) = \{1\}$ or $\{0, 3\}$ for each $v \in V$. Furthermore, he proved that Four-Colors Problem is equivalent to the existence of such a special L -factor.

Naturally, we turn our attention to find a sufficient and necessary condition or a polynomial algorithm for a specific given $L(v)$.

Next, we start with such a case. Let $g : V \rightarrow \mathbb{N}$ and $L(v) = [0, d_G(v)] - \{g(v)\}$. That is, $L(v)$ contains all integers *except exactly one* integer $g(v)$ for each $v \in V(G)$. In this case, L -factor is kind of like a “compliment” of g -factor or “anti-factor”. To be more precise, we look for a subgraph the degree of which is different from a prescribed value $g(v)$ at each vertex v . Clearly, G has such an L -factor if and only if G has a g -avoiding subgraph. This problem can be solved in polynomial time (see [403]). The proof here is based on the original proof in [403].

Theorem 2.4.1 (Lovász [403]). *Let $g : V \rightarrow \mathbb{N}$ and $L(v) = [0, d_G(v)] - \{g(v)\}$. If G contains a cycle, then it must have an L -factor. Moreover, for trees T of order n , then the following three statements are equivalent:*

- (a) T has no L -factors;
- (b) T can be oriented so that the out-degree of v is $g(v)$ for every $v \in V(T)$; and
- (c) $\sum_{v \in V(T)} g(v) = n - 1$, and if any edge is removed from T , both resulting components T_1 and T_2 satisfy $\sum_{v \in V(T_i)} g(v) = |V(T_i)|$ or $|V(T_i)| - 1$ ($i = 1, 2$).

Proof. Let G be a connected graph and g, L be given in the theorem.

Claim 1. For any $x_0 \in V(G)$, there exists a subgraph F such that $d_F(x) \neq g(x)$ for all vertices $x \in V - \{x_0\}$.

Let T_0 be a spanning tree of G . For any vertex x , let e_x be the first edge on the path connecting x to x_0 in T_0 . Let $S_i = \{x \mid d_{T_0}(x, x_0) = i\}$ ($1 \leq i \leq h$). We construct F as follows: at the first step, put into F those edges e_x for which $g(x) = 0$ and $x \in S_h$; ...; in the i th step ($1 < i \leq h$) we put into F those edges e_x for which $x \in S_{h-i+1}$ and x is incident with exactly $g(x)$ edges chosen in the previous steps. Clearly, F satisfies $d_F(x) \neq g(x)$ except possibly for $x = x_0$.

Now, we assume that G contains a cycle $C = x_1x_2 \dots x_k$. From Claim 1, there exists a subgraph F such that $d_F(x) \neq g(x)$ for any $x \in V - \{x_1\}$. Let $F_0 = F - C$. Then $d_{F_0}(x) \neq g(x)$ for any $x \neq x_1, \dots, x_k$. If $d_{F_0}(x_i) \neq g(x_i)$ for $1 \leq i \leq k$, we are done; otherwise, say $d_{F_0}(x_k) = g(x_k)$, then we construct an L -factor step by step: set $F_1 = F_0 \cup x_kx_1$; if F_i has been defined, then let $F_{i+1} = F_i$ if $d_{F_i}(x_i) \neq g(x_i)$ and $F_{i+1} = F_i \cup x_ix_{i+1}$ if $d_{F_i}(x_i) = g(x_i)$. Then F_k is the desired L -factor.

Let T be a tree of order n .

(a) $\Rightarrow$ (c). For any $x_0 \in V$, by Claim 1, there exists a subgraph F such that $d_F(x) \neq g(x)$ for all vertices $x \in V - \{x_0\}$. Since F is not an L -factor, we have $d_F(x_0) = g(x_0)$. Let e_x be the first edge on the path connecting a vertex x to x_0 in T . We claim that $d_F(x) = g(x) - 1$ if $e_x \notin F$ and $d_F(x) = g(x) + 1$ if $e_x \in F$. Otherwise, let y be a vertex failing this property. Without loss of

generality, we assume that y is such a vertex closest to x_0 . Let P be a path connecting y to x_0 . It is easy to verify that the symmetric difference $F \triangle P$ is an L -factor, a contraction.

Therefore,

$$\begin{aligned} \sum_{x \in V} g(x) &= d_F(x_0) + \sum_{\substack{x \\ e_x \in F}} (d_F(x) - 1) + \sum_{\substack{x \\ e_x \notin F}} (d_F(x) + 1) \\ &= \sum_{x \in V} d_F(x) - |F| + (|E(T)| - |F|) \\ &= \sum_{x \in V} d_F(x) + n - 1 - 2|F| = n - 1. \end{aligned}$$

By the same argument, the other statement of (c) follows.

(c) $\Rightarrow$ (b). Let us orient the edge $e = xy$ from y to x if removing e the component T' containing x satisfies $\sum_{x \in T'} g(x) = |T'| - 1$ (and then the other component T'' satisfies $\sum_{x \in T''} g(x) = |T''|$).

For any vertex $x_0 \in T$, let the components of $T - \{x_0\}$ be $C_1, \dots, C_r$. Then

$$\begin{aligned} n - 1 &= \sum_{x \in T} g(x) = g(x_0) + \sum_{i=1}^r \left(\sum_{x \in C_i} g(x) \right) \\ &= g(x_0) + \sum_{i=1}^r |C_i| - (\text{out-degree of } x_0) \\ &= g(x_0) + n - 1 - (\text{out-degree of } x_0). \end{aligned}$$

So, it implies that the out-degree of any vertex x is $g(x)$.

(b) $\Rightarrow$ (a). Let T be a tree of order n which can be oriented with $d^+(x) = g(x)$ for every $x \in T$. Let F be any subgraph of T . We claim that there exists a vertex x_0 so that $d_F(x_0) = g(x_0)$. Use induction on n to show this.

For $n = 2$, clearly it holds.

Assume that the claim holds for a tree of order less than n . It is well-known that for any orientation of a tree, there always exists a source z (i.e., $d^-(z) = 0$). Consider a source z of T . If $d_F(z) = d^+(z)$, then $d_F(z) = g(z)$ and we are done. Otherwise, there exists an arc $zy \notin F$. Removing the arc zy from T , and denote the component containing y by T_y . If T_y is a singleton, i.e., y is a leaf, then $d_F(y) = 0 = d^+(y) = g(y)$. If $|T_y| \geq 2$, by induction hypothesis, there exists a vertex x_0 so that $d_F(x_0) = g(x_0)$ in T_y . Note that for any $x \in T_y$, $d_F(x)$ is the same in T or T_y . Thus x_0 is the desired vertex. $\square$

The above theorem can be restated as a simple sufficient and necessary condition for such L -factors.

Corollary 2.4.1. *Let $L(v) = \mathbb{N} - g(v)$ ($g(v) \in \mathbb{N}$) for any $v \in V(G)$. Then G has an L -factor if and only if G is not a tree which can be oriented so that the out-degree of vertex v is $g(v)$, for all $v \in V(G)$.*

When $L(v)$ contains a “gap” with more than one element, L -factor problem is an NP-complete problem. We state it more precisely: there exists a vertex x such that $L(x) = \{a_0 < \cdots < a_i < \cdots\}$ contains a gap of at least two, i.e., $a_{i+1} \geq a_i + 3$ for some i , Lovász [404] constructed a transformation from Four-Colors Problem to an L -factor problem. Therefore, it is nature to turn the attention to L -factor problems in which $L(v)$ contains only one-element gaps.

Assume that L satisfies the property

$$(*) \quad i \notin L(v) \text{ implies } i + 1 \in L(v), \text{ for every } v \in V(G).$$

Now we follow the war plan described in [403] to present a necessary and sufficient condition for the existence of L -factors satisfying the property (*). Theorem 2.4.2 is the most general result known for L -factor problem and the criterion also is “the” most complex characterization seen in factor theory. The reader needs all patience to go along with us.

Let L be a list function satisfying the property (*) and F any subgraph. For a vertex $x \in V$, if

$$d_F(x) \in L(x), \tag{2.31}$$

then vertex x is called *feasible*. So a subgraph F is an L -factor if and only if every vertex is feasible. We can measure F ’s “deviation” from condition (2.31) by the value

$$def_L(F) = \sum_{x \in V} dist_L(x, F),$$

where $dist_L(x, F)$ is the distance of $d_F(x)$ from the set $L(x)$. Thus, the “solvability” of (2.31) is characterized by the value

$$def_L = \min_F def_L(F).$$

def_L is also called the *total deficiency* of L . Clearly, $def_L = 0$ if and only there exists an L -factor.

A subgraph F is called *L -optimal*, if $def_L(F) = def_L$. Of course, any L -factor is L -optimal. Set

$$I_L(x) = \{d_F(x) \mid F \text{ is an } L\text{-optimal subgraph}\}.$$

For a list L , $\Delta_L(x)$ and $\delta_L(x)$ denote the maximum and the minimum values in the set $L(x)$, respectively. For a vertex set $S \subseteq V$, $\Delta_L(S)$ is the maximum of $\cup_{x \in S} L(x)$ and $\delta_L(S)$ is defined similarly. We now define a vertex partition of $V(G)$ as follow:

$$\begin{aligned}
C_L &= \{x \mid I_L(x) \subseteq L(x)\}, \\
A_L &= \{x \mid \min I_L(x) \geq \Delta_L(x)\}, \\
B_L &= \{x \mid \max I_L(x) \leq \delta_L(x)\}, \\
D_L &= V - A_L - B_L - C_L.
\end{aligned}$$

A graph G is L -critical if it is connected and $V(G) = D_L$.

For any pair of subsets $S, T \subseteq V$ and $S \cap T = \emptyset$, let $\tau_L^1(S, T)$ be the number of those components C_1 of $G - S - T$ which is L_{ST} -critical, i.e., $V(C_1) = D_{L_{ST}}$, where L_{ST} is a list function obtained by restricting L to $G - S - T$; and let $\tau_L^2(S, T)$ be the number of the components C_2 of $G - S - T$ which have no L_{ST} -factors. Moreover, let

$$\nu(S, T) = d_{G-T}(S) + d_{G-S}(T) = \sum_{s \in S} d_{G-T}(s) + \sum_{t \in T} d_{G-S}(t).$$

Theorem 2.4.2 (Lovász [400]). *Let L be a list function satisfying the property (*) and G a graph. Then the total deficiency of G (i.e., the deficiency of an L -optimal subgraph) is*

$$\text{def}(L) = \max_{\substack{S, T \\ S \cap T = \emptyset}} \tau_L^i(S, T) - \nu(S, T) - \Delta_L(S) + \delta_L(T),$$

$i = 1, 2$. Moreover, a graph G has an L -factor if and only if for every $S, T \subseteq V(G)$, $S \cap T = \emptyset$ ($i = 1$ or 2),

$$\tau_L^i(S, T) - \nu(S, T) - \Delta_L(S) + \delta_L(T) \leq 0.$$

Due to the complexity, the above characterization can not be easily used to derive other useful results or apply to related problems. However, in the process of developing the proof, the structures of graphs without L -factors are revealed. In particular, the vertices of the graphs can be partitioned into quadrable (C_L, A_L, B_L, D_L) , similar to the partition (D, A, C) in Gallai-Edmonds Matching Structure Theorem.

Remark 2.4.1. The idea of alternating path is one of the most important and matured techniques in factor theory as it often deduces a polynomial algorithm or a nice sufficient and necessary condition. Most proofs regarding results for 1-factors, f -factors, (g, f) -factors, even some component factors are directly or indirectly based on the modified versions of alternating paths. Are there limitations how far we can apply this technique? It seems that the property (*) is one of them. The alternating arguments fail if factors do not satisfy the property (*). This can be seen clearly in the proof of Theorem 2.4.3.

In general, to find an L -factor or prove its non-existence is a great challenge. However, when property (*) holds, it is surprisingly easy to find a

“nearly-perfect” L -factor, as seen from the next theorem. It is neat and simple, so we include it here for reader’s enjoyment.

Theorem 2.4.3 (Sebö [522]). *Suppose that G is a connected graph and L satisfies the property (*). For any $u \in V(G)$, there exists a subgraph F so that $d_F(x) \in L(x)$ for all $x \in V(G) - \{u\}$.*

Proof. Assume that F is a subgraph of G so that $|\{x \in V(G) - \{u\} \mid d_F(x) \notin L(x)\}|$ is minimized. Suppose, by the contrary, there exists a vertex v , $v \neq u$, with $d_F(v) \notin L(v)$. Let P be a path between u and v , and let $P(x, y)$ denote the subpath between two of its vertices x and y . By (*), then $d_{F \triangle P}(v) \in L(v)$.

Let x be the vertex with $d_{F \triangle P}(x) \notin L(x)$ nearest to v on P (if there exist no such vertices, then define $x = u$). Then $d_{F \triangle P(v, x)}(x) \in L(x)$. Because of the choice of x , we have $d_{F \triangle P(v, x)}(y) = d_F(y) \in L(y)$ for all vertices y between v and x . Thus, v becomes a feasible vertex in $F \triangle P(v, x)$, but $F \triangle P(v, x)$ contradicts the choice of F . $\square$

Theorem 2.4.2 provides a characterization for an L -factor with the property (*), but it does not provide a polynomial algorithm within the proof. Cornuéjols [154] gave a first polynomial algorithm for such L -factors. To avoid the lengthy description and notions, Sebö [522] found a new simpler criterion and a constructive proof, and consequently yielded a polynomial algorithm of complexity $O(|V||E|)$. His algorithm relied on the construction of a sequence called “parity trace”, an L -subgraph can be augmented from a nice parity trace.

A set S is called *odd (even)* if it consists of only odd (even) integers, and a vertex v is *odd (even) parity* if $L(v)$ is odd (even). If S or v is odd or even, we say that it has *fixed parity*.

Denote by V_0^o (V_0^e) the set of odd (even) parity vertices of G , and suppose $\{w_1, \dots, w_k\}$ is an ordering of the vertices in $V - (V_0^o \cup V_0^e)$. Given the sets V_{i-1}^o and V_{i-1}^e , let $\mathcal{C}_i$ be the set of components C of $G - w_i$ such that $C \subseteq V_{i-1}^o \cup V_{i-1}^e$. For such a component C , define $e(w_i, C)$ to be the number of edges joining w_i to some vertex of C .

Let l_i be the number of components $C \in \mathcal{C}_i$ for which $|C \cap V_{i-1}^o|$ is odd. Let t_i be the number of $C \in \mathcal{C}_i$ for which $|C \cap V_{i-1}^o|$ has a different parity from $e(w_i, C)$, and set $u_i = d(w_i) - t_i$. If $[l_i, u_i] \cap L(w_i)$ does not have fixed parity, V_i^o and V_i^e are undefined. If $[l_i, u_i] \cap L(w_i)$ is odd, let $V_i^o = V_{i-1}^o \cup \{w_i\}$ and let $V_i^e = V_{i-1}^e$; if it is even, let $V_i^o = V_{i-1}^o$ and let $V_i^e = V_{i-1}^e \cup \{w_i\}$. (If $[l_i, u_i] \cap L(w_i)$ is empty, it may be viewed as either even or odd.) We call $\{w_1, \dots, w_k\}$ a *parity trace* if the sets V_k^o and V_k^e are defined.

Let us describe the idea behind of parity trace: each vertex w_i in parity trace $\{w_1, \dots, w_k\}$ is a cutvertex; to insure the existence of L -factors in the components of $G - w_i$, it requires l_i edges of $\nabla(w_i)$ to join w_i to the components in $\mathcal{C}_i$ and u_i edges of $\nabla(w_i)$ are “wasted” (i.e., can not be used for other components rather than these in $\mathcal{C}_i$). If $[l_i, u_i] \cap L(w_i) = \emptyset$, then G can

not have L -factors; otherwise, it means that up to this point it is still possible to construct an L -factor. At the end, we either construct an L -factor for G or there does not exist L -factor since the number of odd parity vertices is odd (recall Hand-shaking Theorem), i.e., $|V_k^o|$ is odd.

Example 2.4.1. In Fig. 2.4, $\{a, b\}$ is a parity trace: $l_1 = t_1 = 2$, $u_1 = 3$, $[l_1, u_1] \cap L(a) = 3$, thus $a \in V_1^o$; $l_2 = 2$, $t_2 = 1$, $u_2 = 2$, $[l_2, u_2] \cap L(b) = 2$ and then, $b \in V_2^e$.

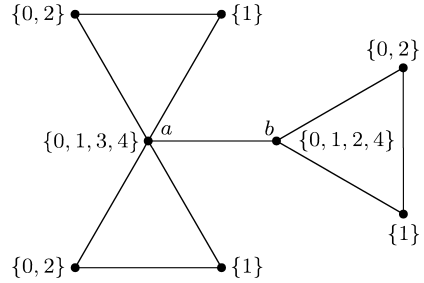


Fig. 2.4 An example of parity trace.

Theorem 2.4.4 (Sebö [522]). Suppose that G is a connected graph and L satisfies the property (*). Then there exists an L -factor if and only if there is no parity trace $\{w_1, \dots, w_k\}$ with $|V_k^o|$ odd.

For a given subset $T \subseteq V(G)$, a subgraph F is called T -join if $d_F(x) \equiv 1 \pmod{2}$ for any $x \in T$ and $d_F(x) \equiv 0 \pmod{2}$ for any $x \in V - T$. The problem of finding a T -join with minimum number of edges is interested to many researchers, due to its close relations with other combinatorial optimization problems (e.g., Chinese Postman Problem).

For smaller T -join, say $|T| = 2$, it is easy to identify all of them. In general, T -join problem can be reduced to the *weighted* matching problem. As an example, we look at the special case of $T = \{x, y\}$. In this case, all T -joins are arisen as the disjoint union of an $x - y$ path and an Eulerian subgraph. So the minimum T -joins are precisely the shortest $x - y$ paths.

Theorem 2.4.5. A connected graph G possesses a T -join if and only if T contains an even number of vertices.

Proof. It is clear to see from Hand-shaking Theorem, that if a T -join exists, then $|T|$ is even.

Conversely, suppose that $|T|$ is even. Begin with $F = \emptyset$, we construct a T -join by adding a path one by one. If there are two vertices $x, y \in T$ with $d_F(x) \equiv d_F(y) \equiv 0 \pmod{2}$, then there is a path P connecting x to y as G is connected. Set $F := F \triangle P$. In the new F , $d_F(x)$ and $d_F(y)$ are odd and all other vertices have the same parities as before. Since $|T|$ is even, after $|T|/2$ steps, we obtain a T -join of G . $\square$

From the above theorem, we see that there is a little requirement for the existence of a T -join. This fact makes T -join a very useful tool for studying many L -factor problems and alike. Furthermore, T -joins also possess a very nice property that one T -join can be transformed to another T -join through a sequence of alternation along cycles. Let F be a T -join and C a cycle. Then $F' = F \triangle C$ is called a T -join from F by a transformation along the cycle C .

Theorem 2.4.6. *Let F_1 and F_2 be two T -joins of a graph G . Then F_2 can be obtained from F_1 by a sequence of transformations along cycles.*

Proof. Let $H = F_1 \triangle F_2$. Then H is a subgraph of G such that the degree of each vertex in H is even and so H contains a cycle C_1 . Let $G_1 = H - E(C_1)$. Then G_1 is still an even graph. If G_1 has no edges, then we have represented H as a union of edge-disjoint cycles. Otherwise, there exists a cycle C_2 in G_1 and by repeating the argument, a new graph G_2 obtained from G_1 by deleting the edge of C_2 . Continue this process, we eventually arrive an empty graph, i.e., H is written as a union of edge-disjoint cycles. By transforming F_1 along each of these cycles in turn, we obtain F_2 . $\square$

Example 2.4.2. *Let $L(v)$ be a set of consecutive odd integers for $v \in T$ and a set of consecutive even integers for $v \in V(G) - T$. Then an L -factor is a T -join, the degrees of which are bounded from above and below. The existence of this special type of L -factors can be reduced to f -factor problems.*

A concept closely related to T -joins is parity graph. A subgraph with all vertices of degree even (resp. odd) is called an *even* (resp. *odd*) subgraph. An even (resp. odd) factor is a spanning even (resp. odd) subgraph F . Clearly, in this case, F is an L -factor satisfying the property (*) and thus its characterization can be derived from Theorem 2.4.2 or Theorem 2.4.3. However, there are some much simpler and direct sufficient conditions and criteria.

Corollary 2.4.2 (see [406]). *If G is of even order, then G has an odd factor; if G is of odd order, then for any vertex $v \in V(G)$, $G - v$ has an odd factor.*

Proof. This is an easy consequence of Theorem 2.4.5 by taking $T = V(G)$ and $T = V(G) - \{v\}$, respectively. $\square$

Corollary 2.4.3 ([130]). *Let $G = (A, B)$ be a connected bipartite graph and L an integer function satisfying $L(x) \in \{1, 3, \dots\}$ for $x \in A$ and $L(y) \in \{0, 2, 4, \dots\}$ for $y \in B$. If $|A|$ is even, then G has an L -factor.*

So, if one bipartition of a bipartite graph G has an even number of vertices, then G has a prescribed L -factor. However, if $|A|$ and $|B|$ are both odd, then G may not have such an L -factor. For example, consider a theta graph $\theta(1, 4m_1 + 1, \dots, 4m_l + 1)$ (the *theta graph* $\theta(\ell_1, \ell_2, \dots, \ell_r)$ is the graph obtained from paths of lengths $\ell_1, \ell_2, \dots, \ell_r$ by identifying each of their end-vertices respectively.)

The even subgraph has been well-studied in graph theory due to its close ties with Eulerian graphs and flow theory. For instance, a graph has a nowhere-zero 4-flow if and only if it is the union of two even graphs. Reader can find many results connected with even graph and its wide applications to other graph theory problems. Here, we restrict ourself to the results related to factor theory only.

Next we give a simple sufficient condition for even factors.

Theorem 2.4.7 (Lovász, see [406]). *Let G be a connected multigraph. If G is 2-edge-connected and $\delta(G) \geq 3$, then G has an even factor.*

Proof. From Petersen's Theorem (i.e., Theorem 1.4.1), if G is 3-regular, then it has a 2-factor. We are done. So we assume that G is not 3-regular.

Let x be a vertex with $d_G(x) \geq 4$.

Claim. There exist $y, z \in N_G(x)$ so that $G - xy - xz$ is connected.

If x is not a cutvertex, i.e., $G - x$ is connected, then for any two edges xy, xz $G - xy - xz$ is still connected; otherwise, x is a cutvertex, let C_1, C_2 be two components of $G - x$ and $y \in V(C_1) \cap N_G(x)$ and $z \in V(C_2) \cap N_G(x)$, then $G - xy - xz$ is connected as G is 2-edge-connected.

We use “splitting process” to transform G to a 3-regular graph: for each vertex x of degree at least four, choose two edges xy and xz incident to x so that $G - xy - xz$ is connected. Splitting x into two vertices x' and x'' , we join x' to x'' , and join two vertices y, z to x' and join all vertices of $N_G(x) - \{y, z\}$ to x'' . Let the resulting graph be G' (i.e. G can be obtained from G' by contracting the edge $x'x''$). It is not hard to check that G' is 2-edge-connected. By repeating this process, we obtain a 2-edge-connected 3-regular graph H in finite steps.

From Theorem 1.4.1, H has a 2-factor F . Contracting all new edges in H to get G , and F is mapped onto a spanning subgraph of G with positive even degrees. $\square$

Theorem 2.4.8 (see [14]). *A graph G has an even factor if and only if for any $S \subset V(G)$,*

$$\sum_{x \in S} (d_G(x) - 2) - \hat{q}(S) \geq 0, \quad (2.32)$$

$\hat{q}(S)$ is the number of components C of $G - S$ such that $e_G(C, S) \equiv 1 \pmod{2}$.

Proof. Set $g(x) = 2$ and $f(x) = N$ for all $x \in V(G)$, where N is a sufficiently large even integer (e.g., letting N even and $N \geq \Delta(G)$). Then G has an even factor if and only G has a parity (g, f) -factor. For any two disjoint subsets S and T , if $S \neq \emptyset$, we have

$$\begin{aligned} \tilde{\delta}(S, T) &= \sum_{x \in S} f(x) + \sum_{x \in T} (d_G(x) - 2) - e_G(S, T) - \tilde{q}(S, T) \\ &\geq N|S| + \sum_{x \in T} (d_G(x) - 2) - e_G(S, T) - \tilde{q}(S, T) \geq 0. \end{aligned}$$

If $S = \emptyset$, then

$$\tilde{\delta}(S, T) = \tilde{\delta}(\emptyset, T) = \sum_{x \in T} (d_G(x) - 2) - \tilde{q}(\emptyset, T) = \sum_{x \in T} (d_G(x) - 2) - \hat{q}(T).$$

Therefore, by Theorem 2.1.3, G has an even factor if and only $\tilde{\delta}(S, T) \geq 0$, i.e., (2.32) holds. $\square$

Now we turn our attention to study L -factors when $L(v)$ takes special values. In many cases, we are able to find a simple sufficient condition or criterion.

Theorem 2.4.9 (Karoński, Łuczak and Thomason [291]). *Let $G = (U, V)$ be a bipartite graph and $0 \leq \alpha \leq 1$. Then G has a spanning subgraph H such that $d_H(u) = \lceil \alpha d_G(u) \rceil$ for $u \in U$ and $d_H(v) \geq \lfloor \alpha d_G(v) \rfloor$ for $v \in V$.*

Proof. At each vertex $u \in U$, color $\lceil \alpha d_G(u) \rceil$ edges red and color the remaining edges blue. Let $r(v)$ be the number of red edges at $v \in V$. If $r(v) < \lfloor \alpha d_G(v) \rfloor$, we call v deficient, and let $D \subset V$ be the set of deficient vertices. We call the quantity $\sum_{v \in D} (\lfloor \alpha d_G(v) \rfloor - r(v))$ the *total deficiency*.

Our aim is to reduce the total deficiency to zero. Suppose $D \neq \emptyset$ and let $v \in D$. If we can find a path $v = v_1 u_1 v_2 u_2 \dots v_k$ such that $v_i u_i$ is blue, $u_i v_{i+1}$ is red and $r(v_k) > \lfloor \alpha d_G(v_k) \rfloor$, then we can recolor $v_i u_i$ red and $u_i v_{i+1}$ blue ($1 \leq i < k$), so reducing the total deficiency. We can keep on repeating this procedure until the total deficiency vanishes. The red edges then form the desired graph H .

Suppose that for some $v \in D$ no such an alternating path exists. Let $A \subset U$ and $B \subset V$ be the set of vertices that can be reached by alternating blue-red paths from v (including length zero so $v \in B$). Then there are no red edges from A to $V - B$, nor are there any blue edges from B to $U - A$. Let E be the set of edges of G between A and B . Since each vertex $u \in A$ has at least $\alpha d_G(u)$ red edges and these all go to B , at least $\alpha|E|$ edges of E are red. But each vertex $w \in B$ has at least $(1 - \alpha)d_G(w)$ blue edges and v has more than $(1 - \alpha)d_G(v)$ blue edges, and all these blue edges go to A , so more than $(1 - \alpha)|E|$ edges of E are blue. This is a contradiction and the proof is complete. $\square$

Next, we investigate odd-factors. Let f be an integer-valued function, $f : V \rightarrow \{1, 3, \dots\}$. A spanning subgraph F is a $(1, f)$ -odd factor if $d_F(x) \in \{1, 3, \dots, f(x)\}$.

The $(1, f)$ -odd factors are a natural extension of 1-factors. It has a surprisingly simple criterion and share many nice properties with 1-factors. In this sense, it is probably the most successful extension of 1-factors. At first, we present the characterization given by Cui and Kano [157]. A special version of this theorem (i.e., $f(x) = k$ for a fixed integer k) was obtained by Amahashi [30].

Theorem 2.4.10 (Cui and Kano [157]). *For an integer-valued function $f : V \rightarrow \{1, 3, \dots\}$, a graph G has a $(1, f)$ -odd factor if and only if for any vertex subset $S \subseteq V$,*

$$o(G - S) \leq f(S). \quad (2.33)$$

Proof. Suppose that G has a $(1, f)$ -odd factor F . If $S = \emptyset$, since every component of F has even order, G has no odd components or $o(G) = 0$, so (2.33) holds.

Let $\emptyset \neq S \subset V(G)$ and C be any odd component of $G - S$. Then there exists an edge of F joining C to S , since every component of F is of even order. Hence

$$o(G - S) \leq e_F(G - S, S) \leq \sum_{x \in S} d_F(x) \leq \sum_{x \in S} f(x),$$

or (2.33) holds.

Conversely, assume that G satisfies (2.33). Let N be a sufficiently large odd integer (e.g., N can be an odd integer greater than $|V(G)| + |E(G)|$). Define $g : V(G) \rightarrow \mathbb{Z}$ by $g(x) = -N$ for all $x \in V(G)$. Then $(1, f)$ -odd factor and parity (g, f) -factor are the same. So it suffices to show that G has a parity (g, f) -factor.

Let S and T be two disjoint subsets of $V(G)$. If $T \neq \emptyset$, then $d_G(x) - g(x) = d_G(x) + N$ for every $x \in T$, where N is sufficiently large, and so

$$\begin{aligned} \tilde{\delta}(S, T) &= \sum_{x \in S} f(x) + \sum_{x \in T} (d_G(x) - g(x)) - e_G(S, T) - \tilde{q}(S, T) \\ &\geq N - e_G(S, T) - \tilde{q}(S, T) \geq 0. \end{aligned}$$

Hence we may assume $T = \emptyset$. It is immediate that any one of $q(S, \emptyset)$ components C of $G - S$ has odd order since $\sum_{x \in V(C)} f(x) \equiv |V(C)| \equiv 1 \pmod{2}$. Hence $q(S, \emptyset) = o(G - S)$, and therefore we have

$$\delta(S, \emptyset) = f(S) - q(S, \emptyset) = f(S) - o(G - S) \geq 0.$$

Consequently, by Parity (g, f) -Factor Theorem, G has a parity (g, f) -factor, which is the desired $(1, f)$ -odd factor of G . $\square$

The $(1, f)$ -odd factor problem has been examined thoroughly, by Kano, Katona, Matsuda, Topp and Vestergaard (see [283], [284], [287], [556]). Many results and techniques paralleling to 1-factors, such as Tutte-Berg formula, alternating path, covering and Gallai-Edmonds Structure Theorem, have been obtained. For details, reader can go to the original articles or the book by Akiyama and Kano [14].

Now we return to the discussion of existence of L -factors when the list set $L(v)$ has rather irregular pattern, i.e, $L(v)$ may not take only even integers, or odd integers, and has the “gap sizes” more than one.

The next result was a conjecture posed in [2] and was first settled in the affirmative by Shirazi and Verstraëte [526]. Their proof is based on Alon’s Combinatorial Nullstellensatz. The shorter and pure graph theory proof given here is due to Frank, Lau and Szabó in [215].

In a directed graph $(V, \vec{E})$ with an orientation D , we denote the in-degree and the out-degree of a vertex $v \in V$ by $d_D^-(v)$ and $d_D^+(v)$, respectively.

Theorem 2.4.11 (Shirazi and Verstraëte [526]). *Let $G = (V, E)$ be a graph and $L : V(G) \rightarrow 2^{\mathbb{N}}$ satisfying*

$$|L(v)| > \lceil \frac{d_G(v)}{2} \rceil$$

for every $v \in V(G)$. Then G has an L -factor.

Proof. It is well-known that every undirected graph G has an orientation D in which

$$d_D^-(v) \geq \lfloor \frac{d_G(v)}{2} \rfloor \quad \text{for every node } v. \quad (2.34)$$

Indeed, by adding a new vertex z to G and joining z to every vertex of G with odd degree, we obtain a graph G^+ in which every degree is even. Hence G^+ is Eulerian, that is, it can decompose into edge-disjoint circuits and therefore it has an orientation in which the in-degree of every vertex equals its out-degree. The restriction of this orientation to G satisfies (2.34).

Let $F(v) = \{0, 1, \dots, d_G(v)\} - L(v)$. Since $|L(v)| > \lceil \frac{d_G(v)}{2} \rceil$, for the given F and the orientation D we have

$$d_D^-(v) \geq \lfloor \frac{d_G(v)}{2} \rfloor \geq (d_G(v)+1) - |L(v)| = |F(v)| \quad \text{for every vertex } v. \quad (2.35)$$

Claim. G has an F -avoiding subgraph.

To show this, we use induction on $|E|$. If 0 is not a forbidden degree at any vertex, then the empty graph $(V, \emptyset)$ is F -avoiding. So, suppose that $0 \in F(t)$ for a vertex t . Then $d_D^-(t) \geq |F(t)| \geq 1$ and hence there exists an edge $e = st$ of G for which $\vec{e}$ is directed toward t . Let $G^- = G - e$ and $D^- = D - \vec{e}$. Define F^- as follows. Let $F^-(t) = \{i - 1 \mid i \in F(t) - \{0\}\}$, $F^-(s) = \{i - 1 \mid i \in F(s) - \{0\}\}$, and for $z \in V - \{s, t\}$ let $F^-(z) = F(z)$. Since $|F^-(t)| = |F(t)| - 1$, $d_{D^-}^-(v) \geq |F^-(v)|$ holds for every vertex v . By induction hypothesis, there is an F^- -avoiding subgraph G'' of G^- . By the construction of F^- , the subgraph $G' := G'' + e$ of G is F -avoiding.

Therefore, G has an L -factor. $\square$

The above proof can be converted into a polynomial algorithm for finding the existence of such L -factors.

The theorem is best possible, there are at least three families of graphs attaining the lower bound of $|L(v)| > \lceil \frac{d_G(v)}{2} \rceil$. Firstly, suppose that G is a bipartite graph and all degrees are even. Let $L(v) = \{0, 1, \dots, d(v)/2\}$ on one part of G and let $L(v) = \{d(v)/2, \dots, d(v)\}$ on the other part, except that for one vertex v_0 we delete the $d(v_0)/2$ from its list. Thus $|L(v)| = d(v)/2 + 1$ for all vertices but one with $|L(v_0)| = d(v_0)/2$. But a simple counting argument shows that there does not exist an L -factor as required. Secondly, let G be a bipartite graph of all degrees odd. Let $L(v) = \{0, 1, \dots, \lfloor d(v)/2 \rfloor\}$ on one part of G and $L(v) = \{\lceil d(v)/2 \rceil, \dots, d(v)\}$ on the other part. Thus $|L(v)| = \lceil d(v)/2 \rceil$ for all v . But again, counting edges shows that there does not exist an L -factor as required. A final example has all degrees even, and $L(v) = \{0, 2, \dots, d(v)\}$ for all vertices except for one vertex v_0 , with $L(v_0) = \{1, 3, \dots, d(v_0) - 1\}$. Again there can not be an L -factor H since $\sum_{x \in V(H)} d_H(x)$ is odd.

The L -factors as a main tool to solve edge-weighting vertex-coloring problem, many interesting results are obtained recently. To demonstrate the techniques and flavor of such approaches, we include the next theorem here for our readers.

Theorem 2.4.12 (Addario-Berry, Dalal and Reed [4]). *Let $G = (V, E)$ be a graph and $a, b : V(G) \rightarrow \mathbb{Z}$ such that for every $v \in V(G)$, $a(v) \leq \lfloor \frac{d(v)}{2} \rfloor \leq b(v) < d(v)$. Define $L(x) = \{a(x), a(x) + 1, b(x), b(x) + 1\}$ for every $x \in V$. If*

$$b(v) \leq \min \left\{ \frac{d(v) + a(v)}{2} + 1, 2a(v) + 3 \right\}, \quad (2.36)$$

then G has an L -factor.

Proof. Suppose that the desired L -factor does not exist. Set $g(v) \in \{a(v), b(v)\}$, $f(v) = g(v) + 1$. By Theorem 2.1.5, there exists a $(0, f)$ -factor F such that the deficiency

$$def(F) = \sum_{v \in V(F)} \max\{0, g(v) - d_F(v)\}$$

is minimized over all choices of $g(v)$, $f(v)$ and $(0, f)$ -factors. Necessarily, there is a vertex $v \in V$ such that $d_F(v) < g(v)$, so the deficiency of F is positive. Moreover, there exist disjoint vertex subsets S and T satisfy (2.5) and the three conditions (a)–(c) in Theorem 2.1.5.

We show the following two claims first.

Claim 1. For any vertex $v \in T$, $g(v) - d_{G-S}(v) \leq d_S(v)/2$.

For any given vertex $v \in T$, we may assume that $d_F(v) < g(v)$ (see Theorem 2.1.5 (c)). If $g(v) \leq d(v)/2$, then

$$g(v) - d_{G-S}(v) \leq \frac{d(v)}{2} - d_{G-S}(v) = \frac{d_S(v) - d_{G-S}(v)}{2} \leq \frac{d_S(v)}{2}$$

holds. So we assume that $g(v) = b(v) > d(v)/2$. Moreover, we have $d_F(v) > a(v) + 1$, otherwise, by setting $g(v) = a(v)$, as $a(v) \leq d(v)/2 < b(v)$, we enable to reduce the deficiency. We may further assume that $d_{G-S}(v) = d_{F-S}(v) > a(v) + 1$, or else by setting $g(v) = a(v)$ and removing $d_F(v) - (a(v) + 1)$ edges from F between v and S , we would reduce the deficiency again. Now, by (2.36), it yields

$$g(v) = b(v) \leq \frac{d(v) + a(v)}{2} + 1 < \frac{d(v) + d_{G-S}(v)}{2} + \frac{1}{2} = d_{G-S}(v) + \frac{d_S(v)}{2} + \frac{1}{2},$$

as $g(v)$ is an integer, so Claim 1 holds.

Claim 2. For any vertex $v \in S$, $f(v) \geq d_T(v)/2$.

For any $v \in S$, we may assume $f(v) = g(v) + 1 < d(v)/2$, otherwise $f(v) \geq d(v)/2 \geq d_T(v)/2$, i.e., the claim holds. Hence $f(v) = a(v) + 1$. Suppose, to the contrary, that the claim fails. Then $2f(v) < d_T(v)$, or $d_T(v) \geq 2a(v) + 3$, and thus $d_T(v) \geq b(v)$ by (2.36). Therefore, there are $d_T(v) - f(v)$ edges from v to T that are not in F — in particular there exists a vertex $w \in N(v) \cap T$ with $vw \notin F$. From part (ii) of Theorem 2.1.5, we can ensure that $d_F(w) < g(w)$ by adopting an appropriate F . (Note that it does not change the fact that $vw \notin F$.) Setting $g(v) = b(v)$ and adding $b(v) - d_F(v) = b(v) - f(v) = b(v) - (a(v) + 1) > 0$ edges from v to T into F (including the edge vw), we decrease the deficiency. Note that we need $b(v) - (a(v) + 1) > 0$ so that we can add edge vw to F to decrease the deficiency. As $b(v) \geq d(v)/2$ and $a(v) + 1 < d(v)/2$, it follows $b(v) - (a(v) + 1) > 0$.

Combining two claims with the fact that $\sum_{v \in T} d_S(v) = \sum_{v \in S} d_T(v)$, we have $\sum_{v \in T} (g(v) - d_{G-S}(v)) \leq f(S)$, a contradiction to (2.5). So we complete the proof. $\square$

From this theorem, we can easily deduce the following one.

Corollary 2.4.4 (Addario-Berry *et al.* [2]). *Let $G = (V, E)$ be a graph and $a, b : V(G) \rightarrow \mathbb{Z}$ such that for every $v \in V(G)$, $\frac{d(v)}{3} - 1 \leq a(v) \leq \frac{d(v)}{2}$ and $\frac{d(v)}{2} - 1 \leq b(v) \leq \frac{2d(v)}{3}$. Define $L(x) = \{a(x), a(x) + 1, b(x), b(x) + 1\}$ for every $x \in V$. Then G has an L -factor.*

Open Problem 2.4.1. *Let $G = (X, Y)$ be a bipartite graph. Can one find a polynomial algorithm for the existence of an L -factor, where $L(x) = \{1\}$ for any $x \in X$ and $L(y) \subseteq N - \{1\}$ for any $y \in Y$?*

A weaker version of this question is as follows: does exist the above mentioned L -factor for every r -regular bipartite graph?

A special case of the above open problem is solved. Consider the component factor problem, $\{K_2, C_3\}$ -factor. This problem, from Hell and Kirkpatrick's Theorem, is a polynomial solvable problem. And a polynomial algorithm is obtained by Cornuéjols, Hartvigsen and Pulleyblank (1982). The existence of such a factor can be formulated into an L -factor problem in a bipartite graph: let G be a graph, $X = V(G)$ and Y be the set of all edges and triangles C_3 in G . Construct a bipartite graph G_0 with bipartition (X, Y) and edges of the form uv with $u \in V(v)$. Then G has a $\{K_2, C_3\}$ -factor if and only if G_0 has an L -factor specified in Open Problem 2.4.1.

Chapter 3

Factors and Graphic Parameters

The graphic parameters play important roles in the study of factors, they are used frequently as sufficient conditions for the existence of factors. Since verifying graphic parameter conditions are often easier than that of characterizations, and as well the parameter conditions reflect the structures and properties of graphs from different perspectives, it is quite common in graph theory to investigate the links among the parameters. In this chapter, we investigate the relationships between graph factors and some popular parameters, such as toughness, binding number, connectivity and minimum degree, etc.

3.1 Toughness and k -factors

In this section, we discuss the existence of k -factors in terms of toughness.

Recall that for a connected non-complete graph G , toughness $t(G)$ is defined as

$$t(G) = \min \left\{ \frac{|S|}{\omega(G-S)} \mid S \subseteq V(G), \omega(G-S) \geq 2 \right\}.$$

If $t(G) \geq k$, we call G k -tough. Hence, if G is k -tough, then for any cut-set $S \subset V(G)$, we have

$$\omega(G-S) \leq \frac{|S|}{t(G)} \leq \frac{|S|}{k} \quad \text{or} \quad |S| \geq k \omega(G-S). \quad (3.1)$$

Recall that $N_G[x] = N_G(x) \cup \{x\}$ and $N_G(S) = \bigcup_{x \in S} N_G(x)$, for $S \subseteq V(G)$.

For the convenience of reference, we restate Tutte's k -Factor Theorem (i.e., Theorem 2.1.1) below. For two disjoint subsets S and T , we use $q_G(S, T)$ to

denote the number of components C of $G - (S \cup T)$ such that $e(V(C), T) + k|C| \equiv 1 \pmod{2}$. Let

$$\delta_G(S, T) = k|S| + \sum_{x \in T} d_{G-S}(x) - k|T| - q_G(S, T).$$

Theorem 3.1.1 (k -Factor Theorem, Tutte [561]).

- (a) G has a k -factor if and only if $\delta_G(S, T) \geq 0$ for all $S, T \subseteq V(G)$ with $S \cap T = \emptyset$;
- (b) $\delta_G(S, T) \equiv k|V(G)| \pmod{2}$ for all $S, T \subseteq V(G)$ with $S \cap T = \emptyset$.

The pair (S, T) is said to be a *minimal pair* if $\delta_G(S, T) < 0$ and either $T = \emptyset$ or for any subset $T' \subset T$, we have $\delta_G(S, T') \geq 0$.

The main theorem for this section is Theorem 3.1.2 due to Enomoto *et al.* [186], which was a conjecture of Chvátal. To prove it, we need a lemma first.

Lemma 3.1.1 (Enomoto *et al.* [186]). *Let G be a graph without k -factor and $k \geq 2$. Suppose (S, T) is a minimal pair of G . If $T \neq \emptyset$, then $\Delta(G[T]) \leq k - 2$.*

Proof. To show this lemma, it is sufficient to prove that $e_G(x, T - \{x\}) \leq k - 2$ for any $x \in T$. Let $T' = T - \{x\}$. Since (S, T) is a minimal pair of G , then $\delta_G(S, T') \geq 0$. Hence by Theorem 3.1.1 (b),

$$\begin{aligned} -2 &\geq \delta_G(S, T) - \delta_G(S, T') \\ &= k|S| + \sum_{y \in T} d_{G-S}(y) - k|T| - q_G(S, T) \\ &\quad - (k|S| + \sum_{y \in T'} d_{G-S}(y) - k|T'| - q_G(S, T')) \\ &= d_{G-S}(x) - k + q_G(S, T') - q_G(S, T). \end{aligned}$$

Since $q_G(S, T') \geq q_G(S, T) - e_G(x, G - (S \cup T))$, so $e_G(x, T') = d_{G-S}(x) - e_G(x, G - (S \cup T)) \leq k - 2$. $\square$

Theorem 3.1.2 (Enomoto *et al.* [186]). *Let G be a graph and $k \geq 2$. If G is k -tough, $|V(G)| \geq k + 1$ and $k|V(G)|$ even, then G has a k -factor.*

Proof. Suppose, to the contrary, that G does not contain k -factors. By Tutte's k -Factor Theorem, there exist two disjoint subsets S and T of $V(G)$ such that $\delta_G(S, T) < 0$. Since the complete graph K_n has a k -factor if and only if $n > k$ and kn is even, and G is connected if and only if its toughness is positive, so in the following, we assume that $k \geq 2$ and G is a connected graph which is not complete. There exists a minimal pair (S, T) because G has no k -factors. Let $U = G - (S \cup T)$, $\{C_1, \dots, C_l\}$ be the set of the connected components of U , where $l = \omega(U)$. We may assume that $N_G(x) = V(G) - \{x\}$ for all

$x \in S$, and every component C_i is complete. Furthermore, let $N = e_G(T, U)$, $M = e_G(T)$, $a = |S|$, $b = |T|$. We have

$$\begin{aligned} 0 &> \delta_G(S, T) = ka + N + 2M - kb - q_G(S, T) \\ &\geq ka + N + 2M - kb - l. \end{aligned} \quad (3.2)$$

Let S_1 be a maximal independent set of T and $T_1 = T - S_1$. For $2 \leq i \leq k-1$, let S_i be a maximal independent set of T_{i-1} and $T_i = T_{i-1} - S_i$ (see Fig. 3.1). By Lemma 3.1.1, $\Delta(G[T]) \leq k-2$ and thus $\Delta(G[T_i]) \leq k-i-2$. Hence T_{k-2} is an independent set, $S_{k-1} = T_{k-2}$, $T_{k-1} = \emptyset$.

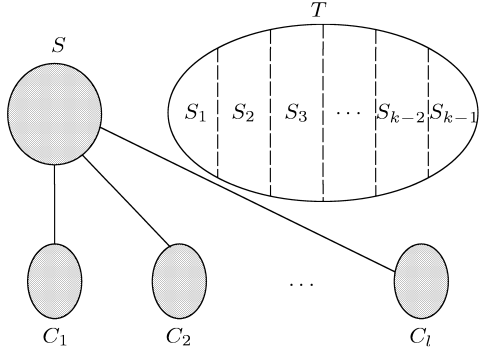


Fig. 3.1 The graph G .

Claim 1. $a + |T_1| + e_G(U, S_1) - l \geq k|S_1|$.

Let $U_1 = \{u \in U \mid e_G(u, S_1) > 0\}$, $L_1 = \{C_i \mid e_G(C_i, S_1) > 0\}$ and $L_2 = \{C_i \mid e_G(u_i, S_1) = 1, \text{ for some } u_i \in V(C_i)\}$. We assume that $L_1 = \{C_1, \dots, C_{l_1}\}$, $L_2 = \{C_1, \dots, C_{l_2}\}$ ($l_2 \leq l_1 \leq l$). Then for each $C_i \in L_2$, choose $u_i \in C_i$ so that $e_G(u_i, S_1) = 1$. Let $U_2 = \{u_i \mid 1 \leq i \leq l_2\}$, $U_3 = U_1 - U_2$. Then $|U_1| \leq e_G(U_1, S_1) - (l_1 - l_2) = e_G(U, S_1) - (l_1 - l_2)$ and $|U_2| = l_2$. Thus $|U_3| \leq e_G(U, S_1) - l_1$. Let $G_1 = G - (S \cup T_1 \cup U_3)$. Since S_1 is an independent set, the vertices of S_1 belong to different components of G_1 . Therefore $\omega(G_1) \geq |S_1| + l - l_1$. We first consider the case of $\omega(G_1) \geq 2$. Because G is k -tough, we have $|S \cup T_1 \cup U_3| \geq k \cdot \omega(G_1) \geq k(|S_1| + l - l_1)$. On the other hand, $|S \cup T_1 \cup U_3| \leq a + |T_1| + e_G(U, S_1) - l_1$. Thus,

$$a + |T_1| + e_G(U, S_1) - l_1 \geq k \cdot \omega(G_1) \geq k(|S_1| + l - l_1) \geq k|S_1| + l - l_1.$$

The claim is proved in this case.

Next consider the case of $\omega(G_1) = 1$. Since $|S_1| + l - l_1 \leq \omega(G_1)$, we have $|S_1| = 0$ or 1 . Suppose that the claim does not hold. Then $a + |T_1| + e_G(U, S_1) - l < k|S_1|$. If $|S_1| = 0$, then $T = \emptyset$. So $T_1 = \emptyset$, $a - l < 0$. If $\omega(G - S) \geq 2$, then $|S| = a \geq kl$, which is clearly impossible. This implies $\omega(G - S) \leq 1$, that is, $l \leq 1$ and $S = \emptyset$. Since G is connected and $k|V(G)| \equiv 0 \pmod{2}$, we have $\delta_k(\emptyset, \emptyset) = 0$, which contradicts to (3.2). Thus, we assume that $|S_1| = 1$ for any

choice of S_1 . It implies that $G[T]$ is complete. Furthermore, $\omega(G_1) = 1$ means that $U \subset N_G(S_1)$. In particular, $e_G(U, S_1) \geq l$ and $a + |T_1| < k|S_1| = k$. Since G is not complete, we have $l \geq 2$. Thus $|S \cup T| \geq k \cdot \omega(U) = kl \geq 2k$. However, $|S \cup T| = a + |T_1| + |S_1| < k + 1$, a contradiction.

Claim 2. For $2 \leq i \leq k-1$, $a + e_G(T - T_{i-1}, S_i) + |T_i| + e_G(U, S_i) \geq k|S_i|$.

If $S_i = \emptyset$, then $T_i = \emptyset$, the claim holds. Now we assume that $S_i \neq \emptyset$, let $X_i = N_G(S_i) \cap (T - T_{i-1})$, $Y_i = N_G(S_i) \cap U$, $G_i = G - N_G(S_i)$. Then S_i is an isolated vertices set of G_i , $\omega(G_i) \geq |S_i|$. We first consider the case of $\omega(G_i) \geq 2$. Then, $|N_G(S_i)| = |S \cup X_i \cup Y_i \cup T_i| \geq k \cdot \omega(G_i) \geq k|S_i|$. On the other hand, $|S \cup X_i \cup Y_i \cup T_i| \leq a + e_G(T - T_{i-1}, S_i) + |T_i| + e_G(U, S_i)$. Thus the claim follows in this case. Next we consider the case of $\omega(G_i) = 1$, that is, $|S_i| = 1$ and $V(G) = N_G(S_i) \cup S_i$. Then

$$|N_G(S_i)| \leq a + e_G(T - T_{i-1}, S_i) + |T_i| + e_G(U, S_i) < k,$$

which contradicts to the assumption that $|V(G)| \geq k + 1$, thus Claim 2 is proved.

By Claims 1 and 2, since $\sum_{i=1}^{k-1} |S_i| = |T| = b$, $T_{k-1} = \emptyset$, $\sum_{i=2}^{k-1} e_G(T - T_{i-1}, S_i) = M$ and $\sum_{i=1}^{k-1} e_G(U, S_i) = N$, we have

$$\begin{aligned} kb &\leq (k-1)a + \sum_{i=1}^{k-1} |T_i| + \sum_{i=2}^{k-1} e_G(T - T_{i-1}, S_i) + \sum_{i=1}^{k-1} e_G(U, S_i) - l \\ &\leq (k-1)a + \sum_{i=1}^{k-2} |T_i| + M + N - l. \end{aligned} \quad (3.3)$$

From (3.2) and (3.3), it yields $(k-1)a + \sum_{i=1}^{k-2} |T_i| + M + N - l > ka + N + 2M - l$ and

$$\sum_{i=1}^{k-2} |T_i| > a + M \geq M. \quad (3.4)$$

On the other hand,

$$M = \sum_{1 \leq i < j \leq k-1} e_G(S_i, S_j) = \sum_{i=1}^{k-2} e_G \left(S_i, \bigcup_{j=i+1}^{k-1} S_j \right) = \sum_{j=1}^{k-2} e_G(S_j, T_j).$$

Since S_j is a maximal independent subset of T_{j-1} , we have $e_G(S_j, x) > 0$ for any $x \in T_j$. Thus $e_G(S_j, T_j) \geq |T_j|$, $M \geq \sum_{j=1}^{k-2} |T_j|$, a contradiction to (3.4). The theorem is proved. $\square$

The toughness condition in Theorem 3.1.2 is sharp as shown in the next theorem.

Theorem 3.1.3 (Enomoto *et al.* [186]). *Let G be a graph with $|V(G)| \geq k+1$ and $k|V(G)|$ even. For any positive real number ε , there exists a $(k-\varepsilon)$ -tough graph G which has no k -factors.*

Enomoto (1986) proved Theorem 3.1.4, which is stronger than Theorem 3.1.2 and showed that the theorem is sharp. To prove Theorem 3.1.4, we need the following lemma.

Lemma 3.1.2 (Enomoto [183]). *Let A and B be two vertex disjoint subsets of $V(G)$ satisfying the following two conditions:*

- (a) $\delta_G(A, B) < 0$;
- (b) *when $A \subset A' \subseteq V(G) - B$, $\delta_G(A', B) \geq 0$.*

Then $e_G(y, B) \leq k-1$ for any vertex y in $V(G) - (A \cup B)$.

Proof. By contradiction. Suppose there exists a vertex y in $V(G) - (A \cup B)$ such that $e_G(y, B) \geq k$. Set $A' = A \cup \{y\}$, then

$$\begin{aligned} \delta_G(A', B) &= k|A'| + \sum_{x \in B} d_{G-A'}(x) - k|B| - q_G(A', B) \\ &\leq k(|A| + 1) + \sum_{x \in B} d_{G-A}(x) - e_G(y, B) - k|B| - (q_G(A, B) - 1) \\ &\leq \delta_G(A, B) + 1. \end{aligned}$$

By Theorem 3.1.1 (b), $\delta_G(A', B) \equiv \delta_G(A, B) \pmod{2}$. Thus $\delta_G(A', B) \leq \delta_G(A, B) < 0$, a contradiction to the maximality of A . $\square$

Theorem 3.1.4 (Enomoto [183]). *Let G be a graph. Suppose $|V(G)| \geq k+1$, $k|V(G)|$ even, and $|S| \geq k \cdot \omega(G-S) - \frac{7}{8}k$ if $\omega(G-S) \geq 2$. Then G has a k -factor.*

Proof. By Theorem 3.1.2, this theorem holds when $k = 1$. In the following, we assume that $k \geq 2$. Let

$$\eta_G(A, B) = k|A| + \sum_{x \in B} d_{G-A}(x) - k|B| - \omega(G - (A \cup B)).$$

From Theorem 3.1.1 and Lemma 3.1.2, it is obvious that we only need to show the following claim.

Claim. Suppose G is not complete. There exist two vertex-disjoint subsets A and B of $V(G)$ satisfying the following conditions:

- (1) $\eta_G(A, B) \leq -2$;
- (2) for any vertex x in A , $N_G(x) = V(G) - \{x\}$;
- (3) for any vertex y in $V(G) - (A \cup B)$, $e_G(y, B) \leq k - 1$.

Then there exists a subset S of $V(G)$ such that $\omega(G - S) \geq 2$ and $|S| < k \cdot \omega(G - S) - \frac{7}{8}k$.

Use induction on $|V(G)|$. First, suppose $|V(G)| \leq k + 1$. Since G is not complete, there exist nonadjacent vertices u and v . Let $S = V(G) - \{u, v\}$. Then $\omega(G - S) = 2$ and $|S| \leq k - 1 < k \cdot \omega(G - S) - \frac{7}{8}k$.

In the following, we assume that $|V(G)| > k + 1$. If there exists a vertex x in G satisfying $d_G(x) \leq k$, then we set $S = N_G(x)$, the claim follows. Thus, for all $x \in V(G)$, we assume that $d_G(x) \geq k + 1$. If $B = \emptyset$, set $S = A$, the claim clearly holds. When $B \neq \emptyset$, let $U = V(G) - (A \cup B)$, $r = \omega(U)$, $\gamma = \min\{d_{G-A}(x) \mid x \in B\}$.

Case 1. $\gamma \geq k$.

From the assumption (1), $r \geq 2$. Let $C_1, \dots, C_r$ be the connected components of U , we choose any vertex y_i ($1 \leq i \leq r$) in the component C_i . Let $R = \bigcup_{i=1}^r N_G(y_i) \cap B$, where I is a maximum independent subset of $B - R$, $S = A \cup R \cup N_G(I)$. Then $\omega(G - S) \geq r + |I| \geq 2$, and

$$\begin{aligned} |S| &\leq |A| + |R| + \sum_{x \in I} d_{G-A}(x) \\ &\leq |A| + r(k - 1) + \sum_{x \in I} (d_{G-A}(x) - k) + k|I| \\ &\leq k(r + |I|) + \eta_G(A, B) - (k - 1)|A| \\ &\leq k \cdot \omega(G - S) - 2 - (k - 1)|A|. \end{aligned}$$

If $A \neq \emptyset$, we have

$$|S| \leq k \cdot \omega(G - S) - k - 1 < k \cdot \omega(G - S) - \frac{7}{8}k.$$

Thus, we assume that $A = \emptyset$. Since $d_G(x) \geq k + 1$ for all $x \in V(G)$, we have $\gamma > k$ and then $|B| - r \leq -2$ by the assumption (1). Thus

$$\omega(G - B) = r \geq |B| + 2 \geq 2, \quad |B| \leq r - 2 < k(r - 1) < k \cdot \omega(G - B) - \frac{7}{8}k.$$

Case 2. $\gamma < k$.

Suppose that $x \in B$ and $d_{G-A}(x) = \gamma$. Choose a subset R of $V(G)$ such that $|R| = k$, $R - A = N_G(x) - A$. Let $G' = G - (R \cup \{x\})$, $A' = A - R$,

$B' = B - (R \cup \{x\})$, $\alpha = |A \cap R|$ and $\beta = |B \cap R|$. Since $k - \beta \geq k - \gamma = \alpha > 0$, so

$$\begin{aligned} \eta_{G'}(A', B') &\leq k(|A| - \alpha) + \sum_{y \in B'} (d_{G-A}(y) - k) - (\omega(U) - |N_G(x) \cap U|) \\ &\leq \eta_G(A, B) - k\alpha - (\beta + 1)(\gamma - k) + (k - \alpha - \beta) \\ &\leq -2 - (k - \beta)(\alpha - 1) \\ &\leq -2. \end{aligned} \tag{3.5}$$

If G' is not complete, then by induction, there exists a subset S' of $V(G')$ such that $\omega(G' - S') > 1$ and $|S'| < k \cdot \omega(G' - S') - \frac{7}{8}k$. Let $S = R \cup S'$. Then

$$\begin{aligned} \omega(G - S) &= \omega(G' - S') + 1 \geq 2, \\ |S| &= |R| + |S'| < k \cdot \omega(G - S) - \frac{7}{8}k. \end{aligned}$$

So we may assume that if $z \in B$ and $d_{G-A}(z) = \gamma$, then $G - (N_G(z) \cup \{z\})$ is complete. Set $\alpha' = |A'|$, $\beta' = |B'|$, $\gamma' = |V(G') - A'|$. From (3.5), it is obvious that $\gamma' > 0$. If $\alpha' < \frac{1}{8}k$, by $\omega(G - (R \cup A')) \geq 2$, $|R \cup A'| < k + \frac{1}{8}k < k \cdot \omega(G - (R \cup A')) - \frac{7}{8}k$, the claim is proved.

Next, we assume that $\alpha' \geq \frac{1}{8}k$. Then from (3.5) we have $\gamma' \leq k - 1$. Note that $\beta \leq \gamma$, $\beta' \leq \gamma'$, and $d_{G-A}(y) \geq \max\{\gamma, \gamma' - 1\}$ for $y \in B'$. Suppose $\gamma > \gamma'$, by the assumption (1), we have

$$\begin{aligned} k\alpha' &\leq -2 - k\alpha + \sum_{z \in B} (k - d_{G-A}(z)) + \omega(U) \\ &\leq -2 - k(k - \gamma) + (\beta + \beta' + 1)(k - \gamma) + (\gamma - \beta + \gamma' - \beta') \\ &= -2 - k(k - \gamma) + (\beta + \beta')(k - \gamma - 1) + k + \gamma' \\ &\leq -2 - k(k - \gamma) + (\gamma + \gamma')(k - \gamma - 1) + k + \gamma' \\ &= -2 + k - k^2 - \gamma + 2k\gamma - \gamma^2 + \gamma'(k - \gamma) \\ &\leq -2 + k - k^2 - \gamma + 2k\gamma - \gamma^2 + (\gamma - 1)(k - \gamma) \\ &= -2 - k^2 + 3k\gamma - 2\gamma^2 \\ &\leq -2 + \frac{1}{8}k^2. \end{aligned}$$

Thus we may assume that $\gamma \leq \gamma'$. From the assumption (1), then

$$\begin{aligned} k\alpha' &\leq -2 - k\alpha + (\beta + 1)(k - \gamma) + \beta'\{k - (\gamma' - 1)\} + \omega(U) \\ &\leq -2 - k(k - \gamma) + (\gamma + 1)(k - \gamma) + \gamma'(k - \gamma' + 1) + 1 \\ &= -2 + k - k^2 + (2k - 1)\gamma - \gamma^2 + \gamma'(k - \gamma' + 1) + 1 \end{aligned}$$

$$\begin{aligned}
&\leq -2 + k - k^2 + (2k - 1)\gamma' - \gamma'^2 + \gamma'(k - \gamma' + 1) + 1 \\
&= -2 + k - k^2 + 3k\gamma' - 2\gamma'^2 + 1 \\
&\leq -1 + k + \frac{1}{8}k^2.
\end{aligned}$$

Thus $\alpha' < 1 + \frac{1}{8}k$. Let $D = R - A$, $D' = V(G') - A'$. If there exists a subset X of D such that $|N_G(X) \cap D'| < |X|$, let $S = (R - X) \cup A' \cup (N_G(X) \cap D')$. Then $\omega(G - S) \geq 2$, and

$$|S| \leq |R| + |A'| - 1 < \frac{9}{8}k \leq k \cdot \omega(G - S) - \frac{7}{8}k.$$

Thus we suppose that $|N_G(X) \cap D'| \geq |X|$ for all subsets X of D . In particular, for all $z \in D$, we have $N_G(z) \cap D' \neq \emptyset$, and $e_G(B', D) \geq |N_G(D) \cap B'| \geq |D| - |D' - B'| = \gamma - \gamma' + \beta'$. Hence

$$\begin{aligned}
\sum_{y \in B'} d_{G-A}(y) &= \sum_{y \in B'} d_{G'-A'}(y) + e_G(B', D) \\
&\geq \beta'(\gamma' - 1) + \gamma - \gamma' + \beta' \\
&= \beta'\gamma' + \gamma - \gamma'.
\end{aligned}$$

Suppose $\beta' < \gamma'$, by the assumption (1), so we have

$$\begin{aligned}
k\alpha' &\leq -2 - k\alpha + (\beta + 1)(k - \gamma) + k\beta' - (\beta'\gamma' + \gamma - \gamma') + (\gamma - \beta + \gamma' - \beta') \\
&\leq -2 - k(k - \gamma) + (\gamma + 1)(k - \gamma) + (\gamma' - 1)(k - \gamma' - 1) - \gamma + 2\gamma' \\
&\leq -2 - k(k - \gamma') + (\gamma' + 1)(k - \gamma') + (\gamma' - 1)(k - \gamma' - 1) + \gamma' \\
&= -1 - k^2 + 3k\gamma' - 2\gamma'^2 \\
&\leq -1 + \frac{1}{8}k^2.
\end{aligned}$$

Hence we may assume that $\beta' = \gamma'$, and $\beta < \gamma$. By the assumption (1), then

$$\begin{aligned}
k\alpha' &\leq -2 - k\alpha + (\beta + 1)(k - \gamma) + k\gamma' - (\gamma'^2 + \gamma - \gamma') + (\gamma - \beta) \\
&\leq -1 - k^2 + (2k - 1)\gamma - \gamma^2 + (k + 1)\gamma' - \gamma'^2 \\
&\leq -1 - k^2 + 3k\gamma' - 2\gamma'^2 \\
&\leq -1 + \frac{1}{8}k^2.
\end{aligned}$$

When $\beta = \gamma$, suppose $d_{G-A}(z) = \gamma$ and $e_G(z, B') = 1$ for some $z \in D$. Then $G - (N_G(z) \cup \{z\})$ is complete, $|D - (N_G(z) \cup \{z\})| = 1$. Let $D - (N_G(z) \cup \{z\}) = \{u\}$. Then $|N_G(u) \cap B'| \geq \gamma' - 1$. This means that

$$\sum_{z \in D} (d_{G-A}(z) - \gamma) + e_G(B', D) \geq \min\{2\gamma, (\gamma - 1) + (\gamma' - 1)\} \geq 2\gamma - 2.$$

Thus

$$\begin{aligned} k\alpha' &\leq -2 - k\alpha + (\gamma + 1)(k - \gamma) + \gamma'(k - (\gamma' - 1)) - (2\gamma - 2) \\ &= k - k^2 + (2k - 3)\gamma - \gamma^2 + (k + 1)\gamma' - \gamma'^2 \\ &\leq k - k^2 + (3k - 2)\gamma' - 2\gamma'^2 \\ &\leq \frac{1}{8}(4 - 4k + k^2) \\ &< \frac{1}{8}k^2. \end{aligned}$$

The claim is proved and thus the proof of the theorem is completed. $\square$

Enomoto [184] introduced a new parameter $\tau(G)$ which is a slight variation of toughness, but seems better to fit the study of factors. For a connected graph G , if it is not complete, we define

$$\tau(G) = \min \left\{ \frac{|S|}{\omega(G - S) - 1} \mid S \subseteq V(G), \omega(G - S) \geq 2 \right\};$$

otherwise, define $\tau(G) = \infty$.

There is a close relationship between $t(G)$ and $\tau(G)$ as shown in the next theorem. In fact, it gives a characterization of toughness in terms of the new parameter $\tau(G)$.

Theorem 3.1.5 (Enomoto [184]). *Suppose G is not complete. Let k be a positive integer, $|V(G)| \geq k + 2$. Then $t(G) \geq k$ if and only if $\tau(G - S) \geq k$ for any subset S of $V(G)$ with $|S| = k$.*

Proof. Suppose $t(G) \geq k$ and let S be a subset of $V(G)$ with $|S| = k$. Then for any subset R of $V(G - S)$, we have $|S \cup R| \geq k \cdot \omega(G - (S \cup R))$, which means that $|R| \geq k(\omega(G - S) - R) - 1$. Thus, $\tau(G - S) \geq k$.

Conversely, let $\tau(G - S) \geq k$ for any subset S of $V(G)$ with $|S| = k$. Let R be a subset of $V(G)$ such that $\omega(G - R) \geq 2$. When $|R| < k$, since $|V(G)| \geq k + 2$, there exists a subset S of $V(G)$ such that $|S| = k$, $R \subset S$, $\omega(G - S) \geq 2$. This means that $\tau(G - S) = 0$, a contradiction. When $|R| \geq k$, we choose an arbitrary subset S of R with $|S| = k$. Then $|R - S| \geq k(\omega(G - R) - 1)$, i.e., $|R| \geq k \cdot \omega(G - R)$. Thus, $t(G) \geq k$. $\square$

Enomoto [184] showed that $\tau \geq 1$ and $\tau \geq 2$ imply the existence of 1-factors and 2-factors, respectively. Therefore, he improved Theorem 3.1.2 for $k = 1, 2$. For general k , the condition $t(G) \geq k$ may not be reduced to $\tau(G) \geq k$ in Theorem 3.1.2, but it is the case if G has sufficiently large number of vertices. The proof is quite long and involved, so we omit it.

Theorem 3.1.6 (Enomoto and Hagita [185]). *Let G be a graph. If $\tau(G) \geq k$, $|V(G)| \geq k^2 - 1$ and $k|V(G)|$ even, then G has a k -factor.*

Several results on factors have appeared relating toughness to (k, m) -factor-critical graphs. A graph G is called (k, m) -factor-critical if $G - S$ has a k -factor for every m -subset S of $V(G)$. Liu and Yu [361] introduced this concept under the name (k, m) -extendable graphs. They proved that if $|V(G)| \geq m + 3$ and $t(G) \geq \max\{3, m\}$, then G is $(2, m)$ -factor-critical. In the same paper, they proposed a conjecture: if $|V(G)| \geq 2m + 1$ and $t(G) \geq m \geq 2$, then G is $(2, 2m - 2)$ -factor-critical. This conjecture is confirmed by Cai, Favaron and Li (1999), and Enomoto (1998), independently.

Theorem 3.1.7 (Cai, Favaron and Li [106]; Enomoto [184]). *Let G be a graph of order n with $t(G) \geq 2$. For every nonnegative integer $m \leq \min\{2t - 2, n - 3\}$, then G is $(2, m)$ -factor-critical.*

For the cases of $k = 1, 3$, Favaron (1996) obtained the following and the proofs can be found in [202], [106] and [525].

Theorem 3.1.8 (Favaron [202]). *Let G be a graph of order n . If m is an integer such that $2 \leq m < n$ and $n + m$ is even, and $t(G) \geq (m + 1)/2$, then G is $(1, m)$ -factor-critical.*

Theorem 3.1.9 (Shi *et al.* [525]). *Let G be a graph of order n with $t(G) \geq 4$. For every nonnegative integer m with $n + m$ even, $m < 2t(G) - 2$ and $m \leq n - 7$, then G is $(3, m)$ -factor-critical.*

Cai *et al.* [106] posed a problem to find conditions on the toughness for a graph to be (k, m) -factor-critical. Enomoto and Hagita (2000) present an answer to this problem when the order of a graph is sufficiently large.

Theorem 3.1.10 (Enomoto and Hagita [185]). *Let G be a graph. Suppose m is a nonnegative integer, k is a positive integer. If $t(G) \geq k$, $t(G) > \frac{m+k-1}{2}$ and $k(|V(G)| - m)$ is even, $|V(G)| \geq m + k^2 - 1$, then G is (k, m) -factor-critical.*

Proof. Choose an arbitrary subset S of $V(G)$ with $|S| = m$. When $m \leq k$, $t(G \vee K_{k-m}) \geq k$, by Theorem 3.1.5, then $\tau(G - S) = \tau(G \vee K_{k-m} - (S \cup K_{k-m})) \geq k$. When $m > k$, let T be any subset of S with $|T| = m - k$. We claim that $t(G - T) \geq k$. Otherwise, assume that there exists a subset U of $V(G - T)$, where $\omega(G - T - U) \geq 2$ and $\frac{|U|}{\omega(G - T - U)} < k$. Then

$$\begin{aligned}
t(G) &\leq \frac{|T \cup U|}{\omega(G - (T \cup U))} \leq \frac{m - k + k \cdot \omega(G - T - U) - 1}{\omega(G - T - U)} \\
&= k + \frac{m - k - 1}{\omega(G - T - U)} \leq k + \frac{m - k - 1}{2} = \frac{m + k - 1}{2},
\end{aligned}$$

a contradiction. By Theorem 3.1.5, $\tau(G - S) = \tau(G - T - (S - T)) \geq k$. In either case, $\tau(G - S) \geq k$, thus by Theorem 3.1.6, $G - S$ has a k -factor. $\square$

Liu and Wang (1998) generalized (k, m) -factor-criticality further to $[a, b]$ -factors. We call a graph G $(a, b; m)$ -factor-critical if for every m -subset $X \subseteq V(G)$ with $|X| = m$, $G - X$ has an $[a, b]$ -factor. Along the same line of discussion above, several sufficient conditions were obtained in [358], but here we only present a criterion of $(a, b; m)$ -factor-criticality.

Theorem 3.1.11 (Liu and Wang [358]). *Let G be a connected graph, and a, b and m be integers such that $1 \leq a < b$ and $1 \leq k$. Then G is $(a, b; m)$ -factor-critical if and only if*

$$\sum_{j=0}^{a-1} (a-j)p_j(G-S) \leq b|S| - bm$$

for all subset $S \subseteq V(G)$ with $|S| \geq m$.

Chen (1991) improved Theorem 3.1.2 by considering k -factors containing a specified edge or excluding a specified edge under the similar condition.

Theorem 3.1.12 (Chen [139]). *Let G be a graph and $k \geq 2$. If $t(G) \geq k$ and $k|V(G)|$ is even, then for every edge e of G , there exists a k -factor which contains the given edge e , and there also exists a k -factor which does not contain e .*

Katerinis and Wang (2008) further extended Theorem 3.1.2 by considering the existence of 2-factors in terms of toughness with inclusion or exclusion property involved two edges.

Theorem 3.1.13 (Katerinis and Wang [299]). *Let G be a 2-tough graph with at least 5 vertices and let e_1, e_2 be a pair of arbitrarily given edges of G . Then*

- (a) *there exists a 2-factor in G containing e_1, e_2 ;*
- (b) *there exists a 2-factor in G avoiding e_1, e_2 ;*
- (c) *there exists a 2-factor in G containing e_1 and avoiding e_2 .*

The toughness conditions for factors can be reduced if the graphs considered are not general graphs. In other words, there are sufficient conditions in terms of toughness for the existence of factors in special classes of graphs.

Here we mentioned one of such a result as an example. Katerinis [297] showed that 1-tough *bipartite* graph has a 2-factor.

Due to the close relationship between 2-factors and the famous Hamiltonian cycles (i.e., connected 2-factor), many researchers have attempted to use t -toughness ($t \leq 2$) combined with other condition, e.g., minimum degree, to insure the existence of 2-factors. We show samples of such results below.

Theorem 3.1.14 (Bauer and Schmeichel [75]). *Let G be a t -tough graph ($1 \leq t \leq 2$) on $n \geq 3$ vertices. If $\delta(G) \geq (\frac{2-t}{1+t})n$, then G has a 2-factor.*

Theorem 3.1.15 (Faudree *et al.* [199]). *Let G be a 1-tough graph on n vertices and $\delta(G) \geq (n-4)/2$. There exists an integer n_0 such that, if $n \geq n_0$, then G has a 2-factor with k cycles, for all k , $1 \leq k \leq (n-10)/4$. In particular, G is Hamiltonian.*

We conclude this section by presenting an open problem.

Open Problem 3.1.1 (see [73]). *Does every $3/2$ -tough maximal planar graph has a 2-factor? Furthermore, does a $3/2$ -tough planar graph has a 2-factor?*

3.2 Toughness and $[a, b]$ -factors

In the previous section, we discussed the relationship between toughness and existence of k -factors. In this section, we extend the discussion to more general factors, $[a, b]$ -factors. In [297], Katerinis generalized Theorem 3.1.2 to $[a, b]$ -factors. Before introducing this pretty result, Theorem 3.2.1, we first review the following definition and lemmas.

Recall that a covering of G is a subset $S \subseteq V(G)$ such that every edge of G has at least one end in S .

Lemma 3.2.1 (Katerinis [297]). *Let $S_1, \dots, S_{a-1}$ be a partition of vertex set of a graph H such that for any vertex x of S_j , $d_H(x) \leq j$ ($S_j = \emptyset$ is allowed). Then there exist a covering set C and an independent set I of H such that*

$$(a-1)c_1 + (a-2)c_2 + \cdots + c_{a-1} \leq (a-1)((a-1)i_1 + (a-2)i_2 + \cdots + i_{a-1}),$$

where $|I \cap S_j| = i_j$, $|C \cap S_j| = c_j$ ($j = 1, \dots, a-1$).

Lemma 3.2.2 (Chvátal [151]). *If G is not a complete graph, then $t(G) \leq \frac{1}{2}\delta(G)$.*

Lemma 3.2.3 (Katerinis [297]). *Let G be a graph and a, b be two positive integers such that $a \leq b$. Suppose that there exist $D, S \subseteq V(G)$, $D \cap S = \emptyset$ such that*

$$\sum_{x \in S} (a - d_{G-D}(x)) > b|D|. \quad (3.6)$$

If S is minimal with respect to (3.6), then for every $x \in S$, we have $d_{G-D}(x) \leq a - 1$.

Now we present the main theorem of this section.

Theorem 3.2.1 (Katerinis [297]). *Let G be a graph and a, b be positive integers such that $a \leq b$. If $t(G) \geq (a - 1) + \frac{a}{b}$ and $a|V(G)| \equiv 0 \pmod{2}$ when $a = b$, then G has an $[a, b]$ -factor.*

Proof. By Theorem 3.1.2, the theorem is true for the case $a = b$, so we may assume that $a < b$ in the following. Suppose, to the contrary, that G does not have $[a, b]$ -factors. Since for every $x \in V(G)$, we have $g(x) = a$, $f(x) = b$, then by Theorem 2.1.4, there exist $D, S \subseteq V(G)$, $D \cap S = \emptyset$ such that

$$\sum_{x \in S} (a - d_{G-D}(x)) > b|D|. \quad (3.7)$$

Let S be minimal with respect to (3.7). Then, by Lemma 3.2.3, $d_{G-D}(x) \leq a - 1$ for every $x \in S$. Define $S_i = \{x \in S \mid d_{G-D}(x) = i\}$ ($0 \leq i \leq a - 1$), $|S_i| = s_i$, $H = G[S_1 \cup S_2 \cup \cdots \cup S_{a-1}]$. Since for every vertex x of S_i , we have $d_H(x) \leq i$, by Lemma 3.2.1, we can find a covering set C and an independent set I of H , such that

$$(a - 1)c_1 + (a - 2)c_2 + \cdots + c_{a-1} \leq (a - 1)((a - 1)i_1 + \cdots + i_{a-1}), \quad (3.8)$$

where $|I \cap S_j| = i_j$, $|C \cap S_j| = c_j$ ($j = 1, \dots, a - 1$). We assume that I is a maximal independent set of H . Let $U = D \cup C \cup (N(I) \cap V(W))$, where $W = (G - D) - S$. Then

$$|U| \leq |D| + \sum_{j=1}^{a-1} j i_j, \quad (3.9)$$

and

$$\omega(G - U) \geq \sum_{j=1}^{a-1} i_j + s_0. \quad (3.10)$$

Let $t(G) = t$. Since G is t -tough, so if $\omega(G - U) \geq 2$, then we have

$$|U| \geq t \cdot \omega(G - U). \quad (3.11)$$

Moreover, by Lemma 3.2.2 and the definition of toughness, every vertex $v \in S$ satisfies that $|U| \geq d_{G-D}(v) + |D| \geq d(v) \geq t + 1$, thus (3.11) holds when $\omega(G - U) = 1$, and when G is complete, we have $\delta(G) \geq t + 1$. Combining (3.9), (3.10) and (3.11), then

$$|D| + \sum_{j=1}^{a-1} j i_j \geq t \sum_{j=1}^{a-1} i_j + t s_0. \quad (3.12)$$

By (3.7), we have $as_0 + \sum_{j=1}^{a-1} (a-j)i_j + \sum_{j=1}^{a-1} (a-j)c_j > b|D|$. Then using (3.12), we obtain

$$as_0 + \sum_{j=1}^{a-1} (a-j)i_j + \sum_{j=1}^{a-1} (a-j)c_j > \sum_{j=1}^{a-1} b(t-j)i_j + bt s_0.$$

Thus $\sum_{j=1}^{a-1} (a-j)c_j > \sum_{j=1}^{a-1} (bt-jb-a+j)i_j$, and since $t(G) \geq (a-1) + \frac{a}{b}$, it follows that $\sum_{j=1}^{a-1} (a-j)c_j > \sum_{j=1}^{a-1} (ba-jb-b+j)i_j$. From (3.8), $\sum_{j=1}^{a-1} (a-1)(a-j)i_j > \sum_{j=1}^{a-1} (ba-jb-b+j)i_j$. But $(a-1)(a-j) \leq (ba-jb-b+j)$ for all j ($1 \leq j \leq a-1$), a contradiction. The proof is completed. $\square$

Another way to generalize Theorem 3.1.2 is to consider f -factors in terms of toughness. The following theorem is one of such results. When $a = b$, Theorem 3.1.2 can be deduced from Theorem 3.2.2.

Theorem 3.2.2 (Katerinis [297]). *Let G be a graph and a, b be two positive integers such that $t(G) \geq \frac{(b+a)^2+2(b-a)}{4a}$ when $b \equiv a \pmod{2}$, or $t(G) \geq \frac{(b+a)^2+2(b-a)+1}{4a}$ when $b \not\equiv a \pmod{2}$. Let $f(x)$ be an integer-valued function defined on $V(G)$ such that $\sum_{x \in V(G)} f(x)$ is even and $a \leq f(x) \leq b$, then G has an f -factor.*

When $a = 2$, Chen (1994) improved Theorem 3.2.1 to the following.

Theorem 3.2.3 (Chen [135]). *Let G be a graph of order at least 3 and $b > 2$. If $t(G) \geq 1 + \frac{1}{b}$, then G has a $[2, b]$ -factor.*

Proof. By contradiction. Suppose that $t(G) \geq 1 + \frac{1}{b}$ and G has no $[2, b]$ -factors. By Theorem 2.1.9, there exists $S \subset V(G)$ such that

$$2p_0(G - S) + p_1(G - S) \geq b|S| + 1, \quad (3.13)$$

where $p_i(G - S)$ is the number of vertices of degree i in graph $G - S$. Without loss of generality, we assume that G is not a complete graph. By Lemma 3.2.2, $\delta(G) \geq 2 + \frac{2}{b} > 2$. We may assume $S \neq \emptyset$ and write $T = \{t \mid t \in V(G) - S, d_{G-S}(t) = 1\}$ and thus $|T| = p_1(G - S)$. We consider two cases.

Case 1. $T = \emptyset$ or $N_{G-S}(T) = T$.

Since $\omega(G - S) \geq p_0(G - S) + \frac{1}{2}p_1(G - S) \geq \frac{1}{2}b|S| + \frac{1}{2} \geq 2$, by (3.13), we have

$$1 + \frac{1}{b} \leq t(G) \leq \frac{|S|}{\omega(G - S)} \leq \frac{|S|}{p_0(G - S) + \frac{1}{2}p_1(G - S)}.$$

Thus

$$2p_0(G - S) + p_1(G - S) \leq \frac{2b}{b+1}|S| < b|S|,$$

which contradicts to (3.13).

Case 2. $T \neq \emptyset$ and $N_{G-S}(T) \neq T$.

There exists a vertex $z \in N_{G-S}(T)$ such that $d_{G-S}(z) \geq 2$. Let r be the number of independent edges of $G - S$. Let $X = S \cup (N_{G-S}(T) - T - \{z\})$ if $|N_{G-S}(\{z\}) \cap T| = 1$, and let $X = S \cup (N_{G-S}(T) - T)$ if $|N_{G-S}(\{z\}) \cap T| > 1$. In either case, we have $|X| \leq |S| + p_1(G - S) - 2r - 1$, and by (3.13), we have $\omega(G - X) \geq p_0(G - S) + (p_1(G - S) - 2r) + r \geq p_0(G - S) + \frac{1}{2}p_1(G - S) \geq \frac{1}{2}b|S| + \frac{1}{2} \geq 2$. From (3.13), we have $2p_0(G - S) + p_1(G - S) > |S|$, hence

$$\begin{aligned} 1 + \frac{1}{b} \leq t(G) &\leq \frac{|X|}{\omega(G - X)} \\ &\leq \frac{|S| + p_1(G - S) - 2r - 1}{p_0(G - S) + (p_1(G - S) - 2r) + r} \\ &\leq \frac{|S| + p_1(G - S) - 1}{p_0(G - S) + p_1(G - S)}. \end{aligned}$$

Thus

$$p_0(G - S) + \frac{1}{b+1}p_1(G - S) \leq \frac{b}{b+1}(|S| - 1).$$

In particular,

$$p_1(G - S) \leq b(|S| - 1).$$

From the inequalities above, it follows that

$$\begin{aligned} 2p_0(G - S) + p_1(G - S) &\leq \frac{2b}{b+1}(|S| - 1) + \frac{b-1}{b+1}p_1(G - S) \\ &\leq \frac{2b}{b+1}(|S| - 1) + \frac{b-1}{b+1}b(|S| - 1) \\ &= b(|S| - 1) < b|S|, \end{aligned}$$

a contradiction. The theorem is proved. $\square$

The following example shows that Theorem 3.2.3 is best possible.

Example 3.2.1. We construct a graph H (see Fig. 3.2), which consists of a K_m , a K_{mb+1} and $(mb+1)$ K_1 . Let $V(K_{mb+1}) = \{x_1, x_2, \dots, x_{mb+1}\}$, $V((mb+1)K_1) = \{y_1, y_2, \dots, y_{mb+1}\}$. With the same subscript j , join each vertex x_j to y_j , and join each vertex y_j to all the vertices of K_m ($1 \leq j \leq mb+1$). It is easy to see that H has no $[2, b]$ -factors. But when m is large enough, $t(H) = \frac{|X|}{\omega(H-X)} = \frac{m+mb}{mb+1} > 1 + \frac{1}{b} - \varepsilon$, where $X = V(K_m) \cup V(K_{mb+1}) - \{x_1\}$ and ε is any positive real number. Hence the toughness condition of Theorem 3.2.3 can not be replaced by $t(G) > 1 + \frac{1}{b} - \varepsilon$.

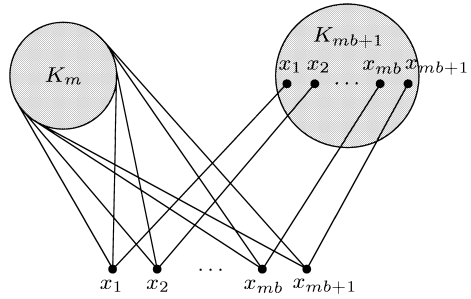


Fig. 3.2 The graph H .

The following results give some sufficient conditions about the existence of $[a, b]$ -factors with restricted conditions by Chen and Liu (1992).

Lemma 3.2.4 (Chen and Liu [140]). Let G be a graph and a, b be integers with $b > a \geq 1$. If $\delta(G) \geq a + 1$ and $\sum_{j=0}^{a-1} (a-j)p_j(G-S) \leq b|S| - 2$ for any non-empty subset S of $V(G)$, then for every edge e of G , there exists an $[a, b]$ -factor which contains the given edge e , and there also exists an $[a, b]$ -factor which does not contain e .

Theorem 3.2.4 (Chen and Liu [140]). Let G be a graph and a, b be integers with $b \geq a \geq 2$. If $t(G) \geq a - 1 + \frac{a}{b}$ and $a|V(G)|$ is even when $a = b$, then for every edge e of G , there exists an $[a, b]$ -factor which contains the given edge e , and there also exists an $[a, b]$ -factor which does not contain e .

Proof. By Theorem 3.1.12, we may assume that $b > a$. When G is complete, by the definition of toughness, we have $\delta(G) = t(G) + 1 \geq a - 1 + \frac{a}{b} + 1 = a + \frac{a}{b}$. Since $\delta(G)$ is an integer, $\delta(G) \geq a + 1$. By Lemma 3.2.2, if G is not complete and $a \geq 2$, $\delta(G) \geq 2t(G) \geq 2(a - 1 + \frac{a}{b}) \geq a + \frac{a}{b}$, or we can say that $\delta(G) \geq a + 1$. By Lemma 3.2.4, it is sufficient for us to prove that for any non-empty subset S of $V(G)$, we have

$$\sum_{j=0}^{a-1} (a-j)p_j(G-S) \leq b|S| - 2. \quad (3.14)$$

For every non-empty subset S of $V(G)$, let $T = \{t \mid t \in V(G) - S, 1 \leq d_{G-S}(t) \leq a-1\}$, $H = G[T]$ and $T_j = \{t \in T \mid d_{G-S}(t) = j\}$, $1 \leq j \leq a-1$ (see Fig. 3.3). Since for every $t \in T_j$, we have $d_H(t) \leq j$, by Lemma 3.2.1, we can find a covering set C and an independent set I of H such that

$$\sum_{j=1}^{a-1} (a-j)c_j \leq (a-1) \sum_{j=1}^{a-1} (a-j)i_j, \quad (3.15)$$

where $|C \cap T_j| = c_j$, $|I \cap T_j| = i_j$ ($j = 1, \dots, a-1$). Clearly, we may assume that I is a maximal independent set of H , $C = V(H) - I$. Then by (3.15), we have

$$\begin{aligned} \sum_{j=0}^{a-1} (a-j)p_j(G-S) &= ap_0(G-S) + \sum_{j=1}^{a-1} (a-j)(c_j + i_j) \\ &\leq ap_0(G-S) + (a-1) \sum_{j=1}^{a-1} (a-j)i_j + \sum_{j=1}^{a-1} (a-j)i_j \\ &= ap_0(G-S) + a \sum_{j=1}^{a-1} (a-j)i_j. \end{aligned} \quad (3.16)$$

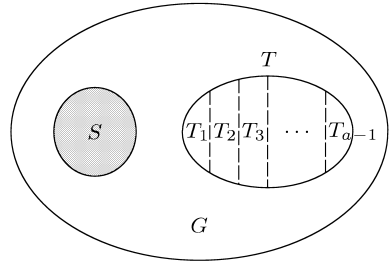


Fig. 3.3 The graph G .

For convenience, we denote $p_0(G-S)$ as p_0 . Now we consider the following cases.

Case 1. $I = \emptyset$.

In this case, $T = \emptyset$. When $\omega(G-S) > 1$, $|S| \geq t(G)\omega(G-S) \geq (a-1 + \frac{a}{b})p_0 \geq (1 + \frac{a}{b})p_0$. Thus $|S| \geq p_0 + 1$. If $\omega(G-S) = 1$, then $p_0 = 0$ or 1 . Obviously, $|S| \geq p_0 + 1$. Thus $b|S| - 2 \geq b(p_0 + 1) - 2 = bp_0 + b - 2$. Then $\sum_{j=0}^{a-1} (a-j)p_j(G-S) = ap_0 \leq bp_0 + b - 2 \leq b|S| - 2$, that is, (3.14) holds.

Case 2. $I \neq \emptyset$.

In this case, either $|I| = 1$ or $N_{G-S}(x) \cap N_{G-S}(y) = \emptyset$ holds for any $x, y \in I$, $x \neq y$. Let $X = S \cup (N_{G-S}(I) - \{z\})$, where $z \in N_{G-S}(I)$. Otherwise,

let $X = S \cup N_{G-S}(I)$. Then

$$|X| \leq |S| + \sum_{j=1}^{a-1} j i_j - 1, \quad (3.17)$$

$$\omega(G - X) \geq \sum_{j=1}^{a-1} i_j + p_0. \quad (3.18)$$

By the definition of toughness, if $\omega(G - X) \geq 2$, we have

$$|X| \geq t(G)\omega(G - X) \geq (a - 1 + \frac{a}{b})\omega(G - X). \quad (3.19)$$

Furthermore, $|X| \geq d_{G-S}(x) + |S| - 1 \geq d_G(x) - 1 \geq \delta(G) - 1 \geq t(G)$ for every $x \in T$. Thus, by Lemma 3.2.2 and the definition of toughness, (3.19) holds when $\omega(G - X) = 1$. Using (3.17), (3.18) and (3.19), we have $|S| + \sum_{j=1}^{a-1} j i_j - 1 \geq$

$(a - 1 + \frac{a}{b})(\sum_{j=1}^{a-1} i_j + p_0)$. Thus, $b|S| \geq \sum_{j=1}^{a-1} (ab - b + a - bj)i_j + (ab - b + a)p_0 + b$.

We want to prove (3.14) by (3.16), it is sufficient to show that

$$ap_0 + a \sum_{j=1}^{a-1} (a - j)i_j \leq \sum_{j=1}^{a-1} (ab - b + a - bj)i_j + (ab - b + a)p_0 + b - 2,$$

that is,

$$\sum_{j=1}^{a-1} (ab - b + a - bj - a^2 + aj)i_j \geq 2 - b - b(a - 1)p_0. \quad (3.20)$$

Since for all j ($1 \leq j \leq a - 1$), $ab - b + a - bj - a^2 + aj = (b - a)(a - 1 - j) \geq 0$ and $2 - b - b(a - 1)p_0 \leq 2 - b < 0$, thus (3.20) holds. The proof is completed. $\square$

When $a = 1$, the following theorem improves Theorem 3.2.4.

Theorem 3.2.5 (Chen and Liu [140]). *Let G be a graph and $b > 1$. If $\delta(G) \geq 2$ and $t(G) > \frac{2}{2b-1}$, then for every edge e of G , there exists a $[1, b]$ -factor which contains the given edge e , and there also exists a $[1, b]$ -factor which does not contain e .*

There are many results on the relation of toughness and other factors as well, for instance, f -factors. Here we mentioned one of such results as the conclusion of this section.

Theorem 3.2.6 (Katerinis [294]).

- (a) Let G be a 2-tough graph and a function $f : V(G) \rightarrow \{1, 2\}$ with $\sum_{x \in (G)} f(x)$ even. Then G has an f -factor;
 (b) Let $1 \leq a \leq b$ be integers. If

$$t(G) \geq \frac{(a+b)2 + 2(b-a)}{4a} \quad \text{if } b \equiv a \pmod{2}$$

or

$$t(G) \geq \frac{(a+b)2 + 2(b-a) + 1}{4a} \quad \text{if } b \not\equiv a \pmod{2},$$

then for every function $f : V(G) \rightarrow \{a, a+1, \dots, b\}$ with $\sum_{x \in V(G)} f(x)$ even, G has an f -factor.

3.3 Binding number and factors

The *binding number* was introduced by Woodall [588] and is defined as

$$\text{bind}(G) = \min \left\{ \frac{|N_G(X)|}{|X|} \mid \emptyset \neq X \subset V(G), N_G(X) \neq V(G) \right\}.$$

Clearly, for any subset S , $\emptyset \neq S \subseteq V(G)$ with $N_G(S) \neq V(G)$, it follows that $|N_G(S)| \geq \text{bind}(G)|S|$.

Binding number plays an important role in the research of graph factors. In 1971, Anderson [54] gave a sufficient condition for the existence of 1-factors in terms of binding number. Later, Woodall (1973) gave a sufficient condition with binding number for the existence of 2-factors.

Theorem 3.3.1 (Anderson [54]). *Let G be a graph of even order. If $\text{bind}(G) \geq \frac{4}{3}$, then G has a 1-factor.*

Theorem 3.3.2 (Woodall [588]). *Let G be a graph. If $\text{bind}(G) \geq \frac{3}{2}$, then G has a 2-factor.*

Next, we present a sufficient condition for a graph to have a regular factor by using binding number, which was obtained by Katerinis and Woodall [303] and by Egawa and Enomoto [169], independently. We first see two lemmas.

Lemma 3.3.1 (Katerinis and Woodall [303]). *Let G be a graph and k a positive integer such that $k|V(G)|$ is even. Suppose that there exist two vertex disjoint subsets S and T of $V(G)$ such that*

$$q_G(S, T) + \sum_{x \in T} (k - d_{G-S}(x)) \geq k|S| + 2. \quad (3.21)$$

Let $W = G - (S \cup T)$. If $|S \cup T|$ is maximal subject to (3.21), then

- (a) for every vertex u of W , we have $d_{G-S}(u) \geq k+1$ and $|N(u) \cap T| \leq k-1$;
 (b) $|C| \geq 3$ for each component C of W , so $|W| \geq 3\omega(W)$;
 (c) when $\omega(W) \geq 3$, we have $2(\text{bind}(G) - 1)\omega(W) \leq |S| + |T|$.

Lemma 3.3.2 (Woodall [588]). *Let G be a graph and $\text{bind}(G) > 0$. Then for each vertex x of G , we have*

$$d_G(x) \geq \frac{(\text{bind}(G) - 1)|V(G)| + 1}{\text{bind}(G)}.$$

Proof. Let $Y = V(G) - N(x)$. Clearly $N(Y) \neq V(G)$ since $x \notin N(Y)$. Then $|N(Y)| \geq \text{bind}(G)|Y|$. Thus $|V(G)| - 1 \geq \text{bind}(G)(|V(G)| - |N(x)|)$, and the lemma follows. $\square$

Theorem 3.3.3 (Katerinis and Woodall [303]; Egawa and Enomoto [169]). *Let G be a graph of order n . If $k \geq 2$, $n \geq 4k - 6$, kn even and $\text{bind}(G) > \frac{(2k-1)(n-1)}{kn-2k+3}$, then G has a k -factor.*

Proof. When $k = 2$, if we can show that $\text{bind}(G) > \frac{3n-3}{2n-1}$, which means $\text{bind}(G) \geq \frac{3}{2}$, then by Theorem 3.3.2, the theorem holds. By the definition of binding number, for any $X \subset V(G)$, we have $n - 1 \geq |N(X)| > \frac{3n-3}{2n-1}|X|$ which implies $|X| < \frac{2n-1}{3}$, thus $|N(X)| > \frac{3n-3}{2n-1}|X| = \frac{3}{2}|X| - \frac{3|X|}{4n-2} > \frac{3}{2}|X| - \frac{1}{2}$, i.e., $|N(X)| \geq \frac{3}{2}|X|$.

We assume that $k \geq 3$. Let $b = \frac{(2k-1)(n-1)}{kn-2k+3}$, thus $\text{bind}(G) > b$. Since $k \geq 3$, we have $b = 2 - \frac{1}{k} + \frac{(2k-1)(k-3)}{k(kn-2k+3)} \geq 2 - \frac{1}{k}$. Thus

$$k(b-1) \geq k-1 \quad \text{and} \quad 2(b-1) > 1. \quad (3.22)$$

Since $n - k \geq k$, we have

$$\frac{(b-1)n+1}{b} - k = \frac{(b-1)(n-k) - (k-1)}{b} \geq 0. \quad (3.23)$$

Suppose, to the contrary, that G has no k -factors. Then by Theorem 3.1.1, there exist two vertex-disjoint subsets S and T of $V(G)$ such that (3.21) holds. Suppose that $|S \cup T|$ is maximal subject to (3.21), so we can use Lemma 3.3.1. Set $W = G - (S \cup T)$ and $\omega = \omega(W)$. Then by (3.21),

$$\omega + \sum_{x \in T} (k - d_{G-S}(x)) \geq k|S| + 2. \quad (3.24)$$

Note that $T \neq \emptyset$, otherwise $\omega \geq k|S| + 2$, which contradicts to Lemma 3.3.1 (c) and (3.22). Let $\delta = \min_{x \in T} \{d_{G-S}(x)\}$. We consider four cases.

Case 1. $\delta = 0$.

Let m_0 be the number of vertices x in T such that $d_{G-S}(x) = 0$. Let $X = T \cup V(W)$. Since $\delta = 0$, then $N(x) \neq V(G)$, $|N(x)| \geq b|X|$. Thus

$$|S| + |T| + |W| - m_0 \geq b(|T| + |W|).$$

That is, $|S| \geq (b-1)(|T| + |W|) + m_0$. By (3.22), $|W| \geq \omega$ and $k > 1$,

$$\begin{aligned} k|S| &\geq (k-1)(|T| + \omega) + km_0 \\ &> (k-1)|T| + \omega + m_0 \\ &\geq \sum_{x \in T} (k - d_{G-S}(x)) + \omega, \end{aligned}$$

a contradiction to (3.24). Thus, $\delta \geq 1$.

Case 2. $\delta \geq k+1$.

Since $T \neq \emptyset$, then (3.24) implies that $\omega \geq k|S| + |T| + 2 \geq 3$. Now by (3.22) and Lemma 3.3.1 (c), it follows that $\omega < 2(b-1)\omega \leq |S| + |T|$, a contradiction. Thus $1 \leq \delta \leq k$.

Case 3. $\delta = k$.

In this case, by (3.24), we have

$$\omega \geq k|S| + 2. \quad (3.25)$$

Let u be a vertex in T such that $d_{G-S}(u) = k$. By Lemma 3.3.2 and (3.23), we have

$$|S| \geq d_G(u) - k > \frac{(b-1)n+1}{b} - k \geq 0.$$

Thus (3.25) implies

$$\omega \geq k+2. \quad (3.26)$$

Let $X = (S \cup V(W)) - N_{G-S}(u)$. Then $|N(X)| \geq b|X|$ since $N(X) \neq V(G)$. Therefore

$$|S| + |T| + |W| - 1 \geq b(|T| + |W| - k).$$

Since $|W| \geq 3\omega$, by Lemma 3.3.1 (b), (3.26) and $b > \frac{3}{2}$, we have

$$\begin{aligned} k|S| &\geq k(b-1)(|T| + |W|) - bk^2 + k \\ &> 3k(b-1)\omega - bk^2 + k \\ &\geq \omega + (3k(b-1) - 1)(k+2) - bk^2 + k \\ &= \omega + k^2(2b-3) + 6k(b-1) - 2 \\ &> \omega, \end{aligned}$$

a contradiction to (3.25).

Case 4. $1 \leq \delta \leq k-1$.

By (3.24), $\omega + (k - \delta)|T| \geq k|S| + 2$. Since $n = |S| + |T| + |W|$, we have

$$(2k - \delta)|T| \geq k(n - |W|) + 2 - \omega. \quad (3.27)$$

Suppose that u is a vertex in T such that $d_{G-S}(u) = \delta$, let $X = (S \cup V(W)) - N_{G-S}(u)$. Then $u \in X - N(X)$, thus $X \neq \emptyset$ and $N(X) \neq V(G)$. We have $|N(X)| > b|X|$ since $\text{bind}(G) > b$. Thus $n - 1 > b(|T| + |W| - \delta)$, that is, $b|T| < n - 1 - b(|W| - \delta)$, combining this with (3.27), we have

$$b(k(n - |W|) + 2 - \omega) < (2k - \delta)(n - 1 - b(|W| - \delta)).$$

Since $\omega \leq |W|$ and $\delta \leq k - 1$, we have

$$b(kn + 2) + (\delta - 2k)(n - 1 + b\delta) < b(\delta - k + 1)|W| \leq 0. \quad (3.28)$$

By the definition of b , $b(kn + 2) - 2k(n - 1) = -(n - 1 - (2k - 1)b)$, thus (3.28) can be simplified into

$$b(\delta - 1) \left(\delta + \frac{n - 1}{b} - (2k - 1) \right) < 0. \quad (3.29)$$

If $2 + \frac{n-1}{b} - (2k - 1) \geq 0$, since $\delta \geq 1$ is an integer, then (3.29) is impossible. This is equivalent to $n \geq 4k + 6$, the proof is completed. $\square$

The bound on binding number in Theorem 3.3.3 is best possible, but the condition on $|G|$ can be improved as shown in the next theorem.

Theorem 3.3.4 (Egawa and Enomoto [169]). *Let G be a graph of order n . If $k \geq 3$, $n > 4k + 1 - 4\sqrt{k} + 2$, kn even and $\text{bind}(G) > \frac{(2k-1)(n-1)}{kn-2k+3}$, then G has a k -factor.*

Chen [132] used binding number to insure the existence of $[2, b]$ -factors and $[3, b]$ -factors. Here, we only present the result about $[2, b]$ -factors. First, we see the following lemmas.

Denote the set of isolated vertices of G by $I(G)$, and let $Q(G) = V(G) - I(G)$ and $q(G) = |Q(G)|$.

Lemma 3.3.3 (Chen [132]). *Let G be a graph and b a positive integer with $b > 2$. Then G has a $[2, b]$ -factor if and only if for all $S \subseteq V(G)$, we have*

$$2i(G - S) + q(G - S) \leq b|S|. \quad (3.30)$$

Proof. The necessary clearly holds.

Conversely, for any $S, T \subseteq V(G)$ and $S \cap T = \emptyset$, by (3.30), we have

$$\begin{aligned} b|S| + \sum_{t \in T} (d_{G-S}(t) - 2) &\geq b|S| + \sum_{t \in T \cap (I(G-S) \cup Q(G-S))} (d_{G-S}(t) - 2) \\ &= b|S| - 2|T \cap I(G - S)| - |T \cap Q(G - S)| \geq 0. \end{aligned}$$

Thus the lemma follows from Theorem 2.1.8. $\square$

Theorem 3.3.5 (Chen [132]). *Let G be a graph of order n and b a positive integer with $b > 2$. If one of the following two conditions holds, then G has a $[2, b]$ -factor.*

- (a) $\text{bind}(G) > 1 + \frac{1}{b}$;
- (b) $\text{bind}(G) = 1 + \frac{1}{b}$ and $bn \equiv 0 \pmod{2}$.

Proof. By Lemma 3.3.3, for any $S \subseteq V(G)$, it is sufficient to prove that (3.30) holds. Clearly, we may assume that $I(G - S) \cup Q(G - S) \neq \emptyset$.

At first, we assume that $\text{bind}(G) = 1 + \frac{1}{b}$ and $bn \equiv 0 \pmod{2}$. Consider two cases.

Case 1. $Q(G - S) = V(G - S)$.

In this case, $i(G - S) = 0$, $q(G - S) \equiv 0 \pmod{2}$. Let v be an arbitrary vertex of $G - S$ and $X = Q(G - S) - \{v\}$. Since $N_G(X) \neq V(G)$, we have

$$1 + \frac{1}{b} = \text{bind}(G) \leq \frac{|N_G(X)|}{|X|} \leq \frac{|S| + q(G - S) - 1}{q(G - S) - 1}.$$

Thus $q(G - S) \leq b|S| + 1$. Moreover, $q(G - S) \leq b|S|$ since $q(G - S) \equiv bn \equiv b|S| \pmod{2}$, and so (3.30) holds.

Case 2. $Q(G - S) \neq V(G - S)$.

Let $X = I(G - S) \cup Q(G - S)$. Since $N_G(X) \neq V(G)$, we have

$$1 + \frac{1}{b} = \text{bind}(G) \leq \frac{|N_G(X)|}{|X|} \leq \frac{|S| + q(G - S)}{i(G - S) + q(G - S)}. \quad (3.31)$$

Thus $(b + 1)i(G - S) + q(G - S) \leq b|S|$ and so (3.30) holds.

Next, suppose that $\text{bind}(G) > 1 + \frac{1}{b}$. If $Q(G - S) \neq V(G - S)$, then (3.31) holds, and (3.30) also holds. Assume that $Q(G - S) = V(G - S)$. Similar to the proof of Case 1, then

$$1 + \frac{1}{b} < \frac{|S| + q(G - S) - 1}{q(G - S) - 1}.$$

Thus $q(G - S) < b|S| + 1$ and so (3.30) holds. The proof is completed. $\square$

Chen (1993) obtained a sufficient condition for $[a, b]$ -factors in terms of binding number as follows. But first we state a result from Woodall (1973).

Lemma 3.3.4 (Woodall [588]). *Let G be a graph. Then, for any independent set X of G , we have*

$$|N_G(X)| \geq \frac{(\text{bind}(G) - 1)|G| + |X|}{\text{bind}(G)}.$$

Theorem 3.3.6 (Chen [133]). *Let a and b be integers such that $2 \leq a < b$, and G a simple graph with $|G| \geq b + 3a$. Set $\lambda = 1 + (a - 1)/b$. If*

$$\text{bind}(G) \geq \lambda \quad \text{and} \quad \delta(G) \geq 1 + \frac{(\lambda - 1)|G|}{\lambda}, \quad (3.32)$$

then G has an $[a, b]$ -factor.

Proof. Suppose that G has no $[a, b]$ -factors. Then, by $[a, b]$ -Factor Theorem, there exist two disjoint subsets S and T of $V(G)$ such that

$$b|S| + \sum_{x \in T} d_{G-S}(x) - a|T| < 0. \quad (3.33)$$

Clearly $T \neq \emptyset$. Define

$$h = \min\{d_{G-S}(x) \mid x \in T\}.$$

Then $0 \leq h \leq a - 1$. We consider the following two cases.

Case 1. $h = 0$.

Let $I = \{x \in T \mid d_{G-S}(x) = 0\}$. Then I is an independent set of G and $I \neq \emptyset$. By Lemma 3.3.4, we have

$$|N_G(I)| \geq \frac{(\lambda - 1)|G| + |I|}{\lambda}. \quad (3.34)$$

On the other hand, by (3.33) we have

$$\begin{aligned} 0 &> b|S| + \sum_{x \in T} d_{G-S}(x) - a|T| \\ &\geq b|S| + |T - I| - a|T| \\ &= b|S| - a|I| + (1 - a)|T - I| \\ &\geq b|S| - a|I| + (1 - a)(|G| - |S| - |I|) \\ &= (a + b - 1)|S| - |I| - (a - 1)|G|. \end{aligned}$$

Thus

$$|N_G(I)| \leq |S| < \frac{|I| + (a - 1)|G|}{a + b - 1} < \frac{|I| + (\lambda - 1)|G|}{\lambda},$$

which contradicts (3.34).

Case 2. $1 \leq h \leq a - 1$.

Since

$$\begin{aligned} 0 &> b|S| + \sum_{x \in T} d_{G-S}(x) - a|T| \geq b|S| + (h - a)|T| \\ &\geq b|S| + (h - a)(|G| - |S|) = (a + b - h)|S| - (a - h)|G|, \end{aligned}$$

we have

$$|S| < \frac{(a-h)|G|}{a+b-h}.$$

By considering a vertex $v \in T$ with $d_{G-S}(v) = h$, we have

$$\delta(G) \leq d_G(v) \leq h + |S| < h + \frac{(a-h)|G|}{a+b-h}.$$

If $h = 1$, then we get a contradiction by the assumption that

$$\delta(G) \geq 1 + \frac{(\lambda-1)|G|}{\lambda} = 1 + \frac{(a-1)|G|}{a+b-1}.$$

So we may assume $h \geq 2$. Let

$$f(h) = h + \frac{(a-h)|G|}{a+b-h} = h + |G| - \frac{b|G|}{a+b-h}, \quad 2 \leq h \leq a-1.$$

Then $f(h)$ reaches its maximum value at $h = 2$ since its derivative

$$\begin{aligned} f'(h) &= 1 - \frac{b|G|}{(a+b-h)^2} \\ &\leq 1 - \frac{b(b+3a)}{(a+b-2)^2} \leq 0 \end{aligned}$$

as $b+3a \leq |G|$ and $2 \leq h$. Hence

$$1 + \frac{(a-1)|G|}{a+b-1} \leq \delta(G) < f(h) \leq f(2) = 2 + \frac{(a-2)|G|}{a+b-2}.$$

This implies $|G| < (a+b-1)(a+b-2)/b < b+3a$, which contradicts $|G| \geq b+3a$. Consequently the theorem is proved. $\square$

The following results give the relationships between the minimum degree, binding number and factors, the readers who are interested in the proofs can go to the references [55] and [554].

Theorem 3.3.7 (Anderson [55]). *Let G be a graph of even order n and c any fixed number. If $\delta(G) \geq cn$, and $\text{bind}(G) > \frac{3-4c}{2-2c}$, then G has a 1-factor.*

Theorem 3.3.8 (Tokushige [554]). *Let G be a graph of order n and c any fixed number. If $\delta(G) \geq cn$, and $\text{bind}(G) > \frac{2-3c}{1-c}$, then G has a 2-factor.*

Theorem 3.3.9 (Tokushige [554]). *Let G be a graph of order n with $k \geq 2$, $n > 4k+1-4\sqrt{k+2}$, kn even and $\text{bind}(G) \geq 2 - \frac{1}{k}$. Assume that $\delta(G) \neq \lfloor \frac{(k-1)n+2k-3}{2k-1} \rfloor$ or $kn \equiv -1, 0, 1, \dots, k-1 \pmod{2k-1}$. Then G has a k -factor.*

The following result gives a sufficient condition about existence of k -factors including or excluding a given edge.

Theorem 3.3.10 (Chen [131]). *Let G be a graph of order n with $k \geq 3$, $n \geq 4k - 6$ and $kn \equiv 0 \pmod{2}$. If $\text{bind}(G) > 2 - \frac{2}{n}$ or $\delta(G) > \frac{n}{2}$, then for any edge e of G , there exists a k -factor which contains the given edge e , and there also exists a k -factor which does not contain e .*

3.4 Connectivity and factors

In this section, we use connectivity as a sufficient condition for the existence of factors. Recall that the independent number, vertex connectivity and edge connectivity are denoted by $\alpha(G)$, $\kappa(G)$ and $\kappa'(G)$, respectively.

We start the investigation with relationships between edge-connectivity and the existence of factors. Let us start with several classic results. The first one is due to Petersen (1891) which is one of the oldest results in graph theory.

Theorem 3.4.1 (Petersen [473]). *Every 2-edge-connected cubic graph has a 1-factor.*

The next result is about 1-factors and the edge-connectivity in regular graphs, which appear in Chapter one (Theorem 1.4.3). We include it here for completeness.

Theorem 3.4.2 (Plesník [476]). *Every $(r - 1)$ -edge-connected r -regular graph of even order has a 1-factor excluding any given $r - 1$ edges of G .*

Note that the above two theorems also hold for multigraphs.

The k -factors in regular graphs is a well-studied topic. In Chapter 1, we showed that every $2r$ -regular graph has a $2k$ -factor for every k , $1 \leq k \leq r$. Note that the complement of a regular-factor in regular graphs is still a regular-factor. For example, the complement of a 1-factor in a r -regular graph is a $(r - 1)$ -factor and the existence of 1-factors in every 2-edge connected cubic graph implies the existence of 2-factors in such a graph. With collective efforts (e.g., Gallai [223], Babler [63], Bollobas *et al.* [85]), the problem of regular factors in regular graphs is completely settled. We summarized all related results in the following theorem. There is an elegant proof given by Akiyama and Kano [14], which treat all cases within a general frame.

Theorem 3.4.3. *Let r and k be integers such that $1 \leq k < r$, and G be an m -edge-connected r -regular graph, where $m \geq 1$. If one of the following conditions holds, then G has a k -factor.*

- (a) r and k are both even; (Petersen [473])
- (b) r is even, k is odd, $|V(G)|$ is even, and $\frac{r}{k} \leq k \leq r(1 - \frac{1}{m})$; (Gallai [223])
- (c) r is odd, k is even and $2 \leq k \leq r(1 - \frac{1}{m})$; (Gallai [223])

- (d) r and k are both odd, and $k \geq \frac{r}{m}$; (Bäbler [63])
 (e) Let m be an integer such that $m^* \in \{m, m+1\}$ and $m^* \equiv 1 \pmod{2}$.
 Either r is odd, k is even and $k \leq r(1 - \frac{1}{m^*})$; or both r and k are odd and
 $k \geq \frac{r}{m^*}$. (Bollobás, Saito and Wormald [85])

Theorem 3.4.3 is sharp for all cases (see examples constructed in [85]).

Corollary 3.4.1 (Bäbler [63]).

- (a) Let $r \geq 3$ be an odd integer. Then every 2-edge-connected r -regular graph has a 2-factor;
 (b) let $r \geq 3$ be an odd integer and k an even integer such that $2 \leq k < r$.
 Then every k -edge-connected r -regular graph has a k -factor.

Remark 3.4.1. The same conditions (a)–(c) can deduce stronger results. Kano [275] showed that under the same conditions in Theorem 3.4.3, for any given edge e , a graph G has a k -factor containing e and another k -factor excluding e .

For regular graphs with edge-connectivity condition, Katerinis (1993) proved a nice result on the existence of all k -factors including and excluding a given set of edges.

Theorem 3.4.4 (Katerinis [298]). Let $1 \leq k \leq r-1$ be integers, and G be a $(r-1)$ -edge-connected r -regular graph of even order. Then for any $r-k$ given edges of G , G has a k -factor excluding these edges; for any given k edges of G , G has another k -factor containing these edges.

To show Theorem 3.4.4, we need two technique lemmas from Katerinis [298].

Lemma 3.4.1 ([298]). Let G be an $(r-1)$ -edge-connected graph and S, T two disjoint subsets of $V(G)$. Let X be the set of any $r-k$ edges of G and $G_1 = G - X$. Then

- (a) $\sum_{x \in T} d_{G_1-S}(x) \geq \sum_{x \in T} d_G(x) - \sum_{x \in S} d_G(x) - 2(r-k);$
 (b) $2 \sum_{x \in T} d_{G_1-S}(x) \geq (r-1) \omega((G_1-S)-T) + \sum_{x \in T} d_G(x) - \sum_{x \in S} d_G(x) - 2(r-k).$

Lemma 3.4.2 ([298]). Let G be a graph with a 1-factor and k an integer. If there exist two disjoint subsets S, T of $V(G)$ such that

$$q_G(S, T) + \sum_{x \in T} (k - d_{G-S}(x)) > k|S|, \quad (3.35)$$

then $|T| \geq |S| + 1$.

Now we are ready to prove Theorem 3.4.4.

Proof of Theorem 3.4.4. Let X be the set of any $r - k$ edges of G and $G_1 = G - X$. If $k = r - 1$, then the theorem holds since, from Theorem 1.4.2 (a), G has a 1-factor containing any given edge and this in turn implies that G has a $(r - 1)$ -factor excluding a given edge. Hence we may assume that $k \leq r - 2$.

Suppose, to the contrary, that G_1 does not have k -factors. Then by k -Factor Theorem (i.e., Theorem 3.1.1) (a) and (b), there exist $S, T \subseteq V(G)$, $S \cap T = \emptyset$ such that

$$\sum_{x \in T} (k - d_{G_1-S}(x)) + q_{G_1}(S, T) \geq k|S| + 2. \quad (3.36)$$

So

$$\omega((G_1 - S) - T) + k(|T| - |S|) - 2 \geq \sum_{x \in T} d_{G_1-S}(x). \quad (3.37)$$

Thus using Lemma 3.4.1 (a) and since G is r -regular, (3.37) implies

$$r|S| \geq r|T| - \omega((G_1 - S) - T) - k(|T| - |S|) + 2 - 2(r - k).$$

Hence

$$\omega((G_1 - S) - T) \geq (r - k)(|T| - |S| - 2) + 2. \quad (3.38)$$

Using Lemma 3.4.1 (a) again, (3.37) implies

$$2(r - k) - 4 \geq (r - 2k)(|T| - |S|) + (r - 3)\omega((G_1 - S) - T). \quad (3.39)$$

By Theorem 3.4.2, G_1 has a 1-factor and since (3.36) holds, using Lemma 3.4.2, then $|T| \geq |S| + 1$. Moreover, since $1 \leq k \leq r - 2$, we have $r \geq 3$.

Suppose that $|T| = |S| + 1$. Then clearly $\omega((G_1 - S) - T) \geq 1$ since $|V(G)|$ is even. So (3.39) implies $2(r - k) - 4 \geq (r - 2k) + (r - 3)$, a contradiction.

Next, suppose that $|T| \geq |S| + 2$, then (3.38) implies

$$\omega((G_1 - S) - T) \geq (r - k - 1)(|T| - |S| - 2) + (|T| - |S| - 2) + 2$$

and so $\omega((G_1 - S) - T) \geq |T| - |S|$. Hence from (3.39) we have $2(r - k) - 4 \geq (r - 2k + r - 3)(|T| - |S|)$, which implies that $-4 \geq -2$ since $k \leq r - 2$ and $|T| \geq |S| + 2$. This contradiction completes the proof of the former result of the theorem.

For the later result, let M be a set of k edges of G and $G_1 = G - M$, then by the result above, G_1 has a $(r - k)$ -factor, say F . Clearly, $G - E(F)$ is a k -regular graph which contains all elements of M . $\square$

Without edge-connectivity condition, the problem of finding k -factors in regular graphs is a challenge one, even to find a regular subgraph (not necessarily spanning) can be a difficult task. For instance, the famous ‘‘Berge’s

Conjecture" (it was proved affirmatively), that every 4-regular graph contains a 3-regular subgraph, is one of such problems. But, if we relax k -factors to $[k, k+1]$ -factors in the conclusion, then it can be more fruitful. The following theorem was conjectured by Erdős and settled by Tutte (1978).

Theorem 3.4.5 (Tutte [565]). *Let k and r be integers such that $1 \leq k \leq r-1$. Then every r -regular graph has a $[k, k+1]$ -factor.*

Theorem 3.4.5 has an interesting generalization from Thomassen (1981) and an elegant proof from him is present below.

Theorem 3.4.6 (Thomassen [545]). *Let k and r be integers such that $1 \leq k \leq r-1$. Then every $[r, r+1]$ -graph has a $[k, k+1]$ -factor.*

Proof. We prove a more general version, i.e., for $[r, r+1]$ -graphs which may have loops and multiple edges.

Clearly, it is sufficient to consider the case $k = r-1$. Assume that G is a *minimal* $[r, r+1]$ -graph, i.e., the deletion of any edge from G results in a vertex of degree less than r . Let A be the set of vertices of degree $r+1$ in G and let B be the set of those vertices of A which are not incident with a loop. By the minimality of G , no two vertices of A are adjacent and, by Hall's Theorem, G contains a set E of independent edges matching B into $V(G) - A$. Let L be a set of loops obtained by selecting one loop from each vertex of $A - B$. Then $G - (E \cup L)$ is an $[r-1, r]$ -graph and the proof is complete. $\square$

From the proof, we see that the above theorem holds for general graphs as well.

There are several generalizations of Theorem 3.4.6, by posing additional sufficient conditions, or finding more general factors (for details, refer to [175], [264], [288], [277]). Below is one of such results.

Theorem 3.4.7 (Kano [277]). *Let k and r be integers such that r odd, $r \geq 3$ and $1 \leq k \leq 2r/3$. Then every r -regular graph has a $[k-1, k]$ -factor such that each of its components is regular.*

Under edge-connectivity, Kano [276] summarized and combined many known results regarding the existence of (g, f) -factors to obtained the following powerful theorem, which included many earlier results as its special cases. The proof here is due to Kano (1985).

Theorem 3.4.8 (Kano [276]). *Let G be an n -edge-connected graph ($n \geq 1$), θ be a real number such that $0 \leq \theta \leq 1$, and $g, f : V(G) \rightarrow \mathbb{Z}^+$ such that $g(x) \leq f(x)$ for all $x \in V(G)$. If one of $\{(ia), (ib)\}$, (ii) and one of $\{(iia), (iib), (iic), (iid), (iie), (iif)\}$ hold, then G has a (g, f) -factor.*

(ia) $g(x) \leq \theta d_G(x) \leq f(x)$ for all $x \in V(G)$;

$$(ib) \ \varepsilon = \sum_{x \in V(G)} (\max\{0, g(x) - \theta d_G(x)\} + \max\{0, \theta d_G(x) - f(x)\}) < 1.$$

(ii) G has at least one vertex v such that $g(v) < f(v)$; or $g(x) = f(x)$ for all $x \in V(G)$ and $\sum_{x \in V(G)} f(x) \equiv 0 \pmod{2}$.

(iiia) $n\theta \geq 1$ and $n(1 - \theta) \geq 1$;

(iiib) both $\{d_G(x) \mid g(x) = f(x), x \in V(G)\}$ and $\{f(x) \mid g(x) = f(x), x \in V(G)\}$ consist of even numbers;

(iiic) $\{d_G(x) \mid g(x) = f(x), x \in V(G)\}$ consists of even numbers, n is odd, $(n+1)\theta \geq 1$ and $(n+1)(1 - \theta) \geq 1$;

(iiid) $\{f(x) \mid g(x) = f(x), x \in V(G)\}$ consists of even numbers and $m(1 - \theta) \geq 1$, where $m \in \{n, n+1\}$ and $m \equiv 1 \pmod{2}$;

(iiie) both $\{d_G(x) \mid g(x) = f(x), x \in V(G)\}$ and $\{f(x) \mid g(x) = f(x), x \in V(G)\}$ consist of odd numbers, and $m\theta \geq 1$, where $m \in \{n, n+1\}$ and $m \equiv 1 \pmod{2}$;

(iiif) $g(x) < f(x)$ for all $x \in V(G)$.

Proof. Let S and T be disjoint subsets of $V(G)$. By (g, f) -Factor Theorem, it is sufficient to show that $\delta(S, T) \geq 0$. By (ii), it follows that $\delta(\emptyset, \emptyset) = 0$, and so we may assume that $S \cup T \neq \emptyset$. Since (ia) implies (ib) with $\varepsilon = 0$, we may assume that (ib), (ii) and one of (iiia)–(iiif) hold. Let $C_1, C_2, \dots, C_q$ be the (g, f) -odd components of $G - (S \cup T)$, where $q = q_G(S, T)$. Then

$$\begin{aligned} \delta(S, T) &= \sum_{x \in S} f(x) + \sum_{x \in T} (d_G(x) - g(x)) - e_G(S, T) - q_G(S, T) \\ &\geq \theta \sum_{x \in S} d_G(x) + (1 - \theta) \sum_{x \in T} d_G(x) - e_G(S, T) - q \\ &\quad - \sum_{x \in S} (\theta d_G(x) - f(x)) - \sum_{x \in T} (g(x) - \theta d_G(x)) \\ &\geq \theta(e_G(S, T) + \sum_{i=1}^q e_G(S, C_i)) + (1 - \theta)(e_G(T, S) \\ &\quad + \sum_{i=1}^q e_G(T, C_i)) - e_G(S, T) - q - \varepsilon \\ &= \sum_{i=1}^q (\theta e_G(S, C_i) + (1 - \theta)e_G(T, C_i) - 1) - \varepsilon. \end{aligned}$$

Since $\delta(S, T)$ is an integer and $\varepsilon < 1$, in order to prove $\delta(S, T) \geq 0$, it suffices to show that every $C = C_i$ satisfies

$$\theta e_G(S, C) + (1 - \theta)e_G(T, C) - 1 \geq 0. \quad (3.40)$$

It is clear that if $e_G(S, C) \geq 1$ and $e_G(T, C) \geq 1$, then (3.40) follows. Hence we may assume that

$$e_G(S, C) = 0 \quad \text{or} \quad e_G(T, C) = 0. \quad (3.41)$$

Since C is a (g, f) -odd components of $G - (S \cup T)$ and G is n -edge-connected, we have $g(x) = f(x)$ for all $x \in V(C)$ and

$$\sum_{x \in V(C)} f(x) + e_G(C, T) \equiv 1 \pmod{2} \quad \text{and} \quad e_G(S \cup T, C) \geq n. \quad (3.42)$$

Moreover, we have

$$\sum_{x \in V(C)} d_G(x) = 2|E(C)| + e_G(C, S \cup T). \quad (3.43)$$

Assume that (iiia) holds. Then by (3.41) and (3.42), we have either $e_G(T, C) = 0$ and $e_G(S, C) \geq n$ or $e_G(S, C) = 0$ and $e_G(T, C) \geq n$, and thus

$$\theta e_G(S, C) - 1 \geq \theta n - 1 \geq 0 \quad \text{or} \quad (1 - \theta)e_G(T, C) \geq (1 - \theta)n - 1 \geq 0.$$

Hence (3.40) follows.

Assume (iiib) holds. By (3.42), $e_G(C, T) \equiv 1 \pmod{2}$. Hence $e_G(C, T) \geq 1$, in particular, we may assume $e_G(S, C) = 0$ by (3.41). So, by (3.43) we have $e_G(C, T) \equiv 0 \pmod{2}$, which contradicts the above $e_G(C, T) \equiv 1 \pmod{2}$. Thus (3.40) follows.

Assume (iiic) holds. By (3.43), we have $e_G(C, S \cup T) \equiv 0 \pmod{2}$, which implies $e_G(C, S \cup T) \geq n + 1$ as n is odd. So if $e_G(C, T) = 0$, then $\theta e_G(S, C) - 1 \geq \theta(n + 1) - 1 \geq 0$. If $e_G(C, S) = 0$, then $(1 - \theta)e_G(S, T) - 1 \geq (1 - \theta)(n + 1) - 1 \geq 0$. Hence (3.40) follows.

Assume (iiid) holds. By (3.42), we have $e_G(C, T) \equiv 1 \pmod{2}$. We may assume $e_G(C, S) = 0$. Then $e_G(T, C) = e_G(S \cup T, C) \geq n$ and so $e_G(T, C) \geq m$, where m is defined in (iiid). Hence $(1 - \theta)e_G(T, C) - 1 \geq (1 - \theta)m - 1 \geq 0$. Therefore (3.40) follows.

Assume (iiie) holds. By (3.42), $|C| + e_G(C, T) \equiv 1 \pmod{2}$. By (3.43), we have $|C| \equiv e_G(C, S \cup T) \pmod{2}$. Thus we have a contradiction when $e_G(S, C) = 0$. Hence $e_G(T, C) = 0$. Then $e_G(S, C) \equiv |C| \equiv 1 \pmod{2}$. Therefore $\theta e_G(S, C) - 1 \geq \theta m - 1 \geq 0$, and (3.40) follows.

Assume (iiif) holds. In this case $q = q_G(S, T) = 0$ and so $\delta(S, T) \geq -\varepsilon$ from the first calculation. Consequently the theorem is proved. $\square$

Next, we discuss the existence of factors in terms of vertex connectivity and other parameters.

Theorem 3.4.9 (Schönberger [517]). *Let G be a 2-connected cubic multi-graph. Then G has a 1-factor which contains any given edge, and for any two edges in G , there exists a 1-factor which does not contain the two edges.*

If G does not contain the induced subgraphs isomorphic to the complete bipartite graph $K_{1,3}$, then G is said to be *claw-free*, or G is called a $K_{1,3}$ -free graph. Similarly, we call a graph $K_{1,n}$ -free if it contains no induced subgraphs isomorphic to $K_{1,n}$.

Theorem 3.4.10 (Plummer [488]). *Let G be a 3-connected claw-free graph of even order. Then G is bicritical.*

Theorem 3.4.11 (Kaneko and Yoshimoto [274]). *Let G be a k -connected graph of at least 6 vertices and $\alpha(G) \leq k$ ($k \geq 4$). Then G has a 2-factor with exactly two components.*

Katerinis first used the vertex connectivity and independent number to discuss the existence of k -factors in graphs, and obtained the following.

Theorem 3.4.12 (Katerinis [293]). *Let G be a graph and $k \geq 2$ be an integer with $k|V(G)|$ even. If $|V(G)| \geq k+1$ and $\kappa(G) > \frac{(k+1)^2}{4k}\alpha(G) + \frac{5k-4}{8} - \frac{2}{k}$, then G has a k -factor.*

Tsuyoshi [559] partially generalized Theorem 3.4.12 to Theorems 3.4.13 and 3.4.14; Niessen [446] generalized Theorem 3.4.12 to Theorem 3.4.15 in a different direction. But neither implies each other.

Theorem 3.4.13 (Tsuyoshi [559]). *Let G be a graph of even order and $k \geq 1$ be an odd integer. If $\kappa(G) \geq \frac{(k+1)^2}{2}$ and $(k+1)^2\alpha(G) \leq 4k\kappa(G)$, then G has a k -factor.*

Proof. By contradiction. Let G be a graph satisfying the hypothesis, but G has no k -factors. Then by Theorem 3.1.1, there exist two vertex disjoint subsets S and T of $V(G)$ such that $\delta(S, T) < 0$. We choose subsets S and T such that $|S \cup T|$ is as large as possible. We have the following claim for those S and T .

Claim. The order of each component C of $G - (S \cup T)$ is greater than 1.

Otherwise, there exists a component C with $|C| = 1$, say $C = \{v\}$. If $e_G(v, T) \geq k$, let $S' = S \cup \{v\}$, we have

$$\begin{aligned} \delta(S', T) &= k|S'| + \sum_{x \in T} d_{G-S'}(x) - k|T| - q(S', T) \\ &\leq k(|S| + 1) + \sum_{x \in T} d_{G-S}(x) - e_G(v, T) - k|T| - (q(S, T) - 1) \\ &\leq \delta(S, T) + 1. \end{aligned}$$

Because $\delta(S', T) \equiv \delta(S, T) \pmod{2}$ and $\delta(S, T) < 0$, so $\delta(S', T) < 0$, which contradicts to the maximality of $|S \cup T|$. If $e_G(v, T) \leq k-1$, let $T' = T \cup \{v\}$, then

$$\begin{aligned} \delta(S, T') &= k|S| + \sum_{x \in T'} d_{G-S}(x) - k|T'| - q(S, T') \\ &\leq k|S| + \sum_{x \in T} d_{G-S}(x) + e_G(v, T) - k(|T| + 1) - (q(S, T) - 1) \\ &\leq \delta(S, T) < 0. \end{aligned}$$

Similarly, this is a contradiction to the maximality of $|S \cup T|$. The claim is proved.

For convenience, let $U = G - (S \cup T)$ and $\{C_1, \dots, C_l\}$ be the set of connected components of U , where $l = \omega(U)$. Now we consider the subgraph $G[T]$ of G induced by T . Let $T_1 = G[T]$ and x_1 be the vertex in T_1 with minimum degree, and $N_1 = N_{T_1}[x_1]$. For $i \geq 2$, let x_i be a vertex with the minimum degree in $T_i = G[T] - \bigcup_{1 \leq j \leq i-1} N_j$, and $N_i = N_{T_i}[x_i]$ (see Fig. 3.4). Denote the order of N_i by n_i . Repeat the above procedures until we reach the situation in which $T_i = \emptyset$ for some i , say for $i = m+1$. Then we have a sequence of vertex sets $\{N_1, \dots, N_m\}$. Obviously, we have the following three properties:

$$\alpha(G[T]) \geq m, \quad (3.44)$$

$$|T| = \sum_{1 \leq i \leq m} n_i, \quad (3.45)$$

$$\sum_{1 \leq i \leq m} \left(\sum_{x \in N_i} d_{T_i}(x) \right) \geq \sum_{1 \leq i \leq m} (n_i^2 - n_i), \quad (3.46)$$

where (3.46) follows because the choice of x_i , that is, all vertices in N_i have degrees at least $n_i - 1$ in T_i .

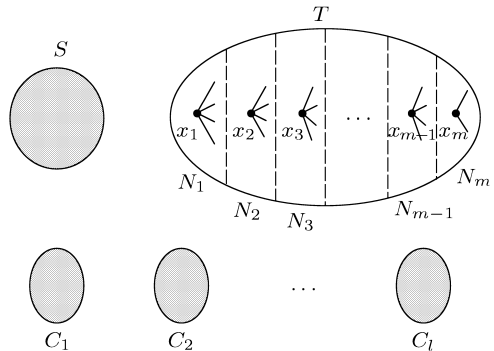


Fig. 3.4 Illustration of the proof.

Set $t = \kappa(G - S)$, we shall prove the following inequality:

$$\sum_{x \in T} d_{G-S}(x) \geq \sum_{1 \leq i \leq m} (n_i^2 - n_i) + \frac{(mt + e_G(T, U))}{2}. \quad (3.47)$$

On the left hand side of (3.46), we calculate the edge joining a vertex x in N_i and a vertex y in N_j only once, that is, we only calculate $d_{T_i}(x)$ but not $d_{T_j}(y)$, thus

$$\sum_{x \in T} d_{G-S}(x) \geq \sum_{1 \leq i \leq m} (n_i^2 - n_i) + \sum_{1 \leq i \leq j \leq m} e_G(N_i, N_j) + e_G(T, U). \quad (3.48)$$

On the other hand, since $\kappa(G - S) = t$, for every N_i ($1 \leq i \leq m$) we have $e_G(N_i, \bigcup_{j \neq i} N_j) + e_G(N_i, U) \geq t$. (When $m = 1$ and $U = \emptyset$, we do not have this inequality. In that case, by the choice of x_1 , we have $\delta(S, T) = k|S| + n_1(n_1 - 1) - kn_1 < 0$, so we conclude that $|V(G)| = |S| + n_1 < -\frac{n_1(n_1-1)}{k} + 2n_1 = \frac{n_1(2k+1-n_1)}{k} \leq \frac{k(k+1)}{k} = k+1$, which is a contradiction to $|V(G)| > \kappa(G) \geq \frac{(k+1)^2}{2}$.) For all i ($1 \leq i \leq m$), summing up all the inequalities, we have

$$\begin{aligned} & \sum_{1 \leq i \leq m} \left(e_G \left(N_i, \bigcup_{j \neq i} N_j \right) + e_G(N_i, U) \right) \\ &= 2 \sum_{1 \leq i < j \leq m} e_G(N_i, N_j) + e_G(T, U) \geq mt. \end{aligned} \quad (3.49)$$

Now (3.47) is implied by (3.48) and (3.49).

It is clear that we have $n_i(n_i - k - 1) \geq -\frac{(k+1)^2}{4}$ and $q(S, T) \geq \omega(U)$, and by (3.45) and (3.47), we have

$$\begin{aligned} 0 > \delta(S, T) &= k|S| + \sum_{x \in T} d_{G-S}(x) - k|T| - q(S, T) \\ &\geq k|S| + \sum_{1 \leq i \leq m} (n_i(n_i - k - 1)) + \frac{mt + e_G(T, U)}{2} - \omega(U) \\ &\geq k|S| - \frac{(k+1)^2 m}{4} + \frac{mt + e_G(T, U)}{2} - \omega(U). \end{aligned} \quad (3.50)$$

Now we estimate $e_G(T, U)$. If $t \geq 1$, then $T \neq \emptyset$, because if $t \geq 1$ and $T = \emptyset$, then $\omega(U) = 1$, which forces $q(S, T) = 1$ and $S = \emptyset$ since $\delta(S, T) < 0$, which is impossible since $|V(G)|$ is even. In particular, for all C_i 's, we have $e_G(C_i, T) \geq t$. By the claim, all C_i satisfy the condition $C_i \subseteq N_G(\{x_1, \dots, x_m\})$, thus $e_G(C_i, T) \geq 2$ (even if $t \leq 1$). We denote a and b the numbers of C_i such that $C_i \subseteq N_G(\{x_1, \dots, x_m\})$ and $C_i \not\subseteq N_G(\{x_1, \dots, x_m\})$ respectively, then

we have

$$e_G(T, U) \geq 2a + tb. \quad (3.51)$$

Obviously, we have

$$\omega(U) = a + b, \quad (3.52)$$

$$\alpha(G) \geq m + b, \quad (3.53)$$

$$\kappa(G) \leq |S| + t. \quad (3.54)$$

By (3.50), (3.51) and (3.52),

$$\begin{aligned} 0 &> k|S| - \frac{(k+1)^2 m}{4} + \frac{mt + e_G(T, U)}{2} - \omega(U) \\ &\geq k|S| - \frac{(k+1)^2 m}{4} + \frac{mt}{2} + \frac{bt}{2} - b \\ &\geq k|S| - \frac{(k+1)^2(m+b)}{4} + \frac{(m+b)t}{2}. \end{aligned} \quad (3.55)$$

So we conclude that $\frac{(m+b)t}{2} - \frac{(k+1)^2(m+b)}{4} < 0$, that is, $2t < (k+1)^2$. By (3.53) and (3.54), we may replace $m+b$ by $\frac{4k\kappa(G)}{(k+1)^2}$ and $|S|$ by $\kappa(G) - t$ in (3.55), then

$$0 > k(\kappa(G) - t) - k\kappa(G) + \frac{2kt\kappa(G)}{(k+1)^2} = kt \left(\frac{2\kappa(G)}{(k+1)^2} - 1 \right) \geq 0,$$

a contradiction. So we complete the proof. $\square$

Theorem 3.4.14 (Tsuyoshi [559]). *Let G be a graph and $k \geq 2$ be even. If $\kappa(G) \geq \frac{k(k+2)}{2}$ and $(k+2)\alpha(G) \leq 4\kappa(G)$, then G has a k -factor.*

Theorem 3.4.15 (Niessen [446]). *Let k be a positive integer and let G be a graph of order n with $n \geq k+1$ and kn even. If G satisfies*

$$\kappa(G) > \begin{cases} \frac{k+2}{4}\alpha(G) + \frac{5k-3}{8} - \frac{2}{k} & \text{if } k \text{ is even,} \\ \frac{(k+1)^2}{4k}\alpha(G) + \frac{5k-4}{8} - \frac{2}{k} & \text{if } k \text{ is odd.} \end{cases}$$

Then G has a k -factor.

A graph G is called a (g, f) -graph if itself is a (g, f) -factor. In the following, we present several sufficient conditions in terms of vertex connectivity

regarding the existence of (g, f) -factor with special properties in bipartite (mg, mf) -graph. The proofs can be found in [366] and [80]. Recall that $\beta^*(G)$ denote the vertex covering number of G .

Theorem 3.4.16 (Liu and Zang [366]). *Let $G = (X, Y)$ be a bipartite (mk) -regular graph and D a set of $E(G)$ such that $|D| = mk - k + r + 1$, $r \geq 0$. If $\kappa(G) \geq \beta^*(G[D]) + r + k$ and $\delta(G - D) \geq k$, then $G - D$ has a k -factor.*

Theorem 3.4.17 (Bian [80]). *Let $G = (X, Y)$ be a bipartite (mg, mf) -graph and D a set of $E(G)$ such that $|D| = m + r$, $r \geq 0$. Let $k = \min\{g(x) \mid x \in V(G)\}$. If*

- (a) $f(X) = f(Y)$, $g(X) = g(Y)$;
- (b) $mk \geq \kappa(G) \geq \beta^*(G[D]) + r + k$;
- (c) $|D \cap E(x)| \leq (m - 1)k$ for all $x \in V(G)$,

where $E(x)$ is the set of edges incident with vertex x in G . Then $G - D$ has a (g, f) -factor.

In Theorem 3.4.17, let $g(x) = f(x) = 1$ for all $x \in V(G)$, we have the following interesting consequences.

Corollary 3.4.2 (Katerinis and Tsikopoulos [300]). *Let $G = (X, Y)$ be an m -regular graph and D a set of $E(G)$ such that $|D| = m + r$, $r \geq 0$. If $\kappa(G) \geq \beta^*(G[D]) + r + 1$ and $\delta(G - D) \geq 1$, then $G - D$ has a perfect matching.*

A graph G is called an f -graph if G itself is an f -factor. Then we have the following result. The proof can be found in [366].

Corollary 3.4.3 (Liu and Zang [366]). *Let $G = (X, Y)$ be a bipartite (mf) -graph and D a set of $E(G)$ such that $|D| = m + r$, $r \geq 0$. If $\kappa(G) \geq \beta^*(G[D]) + r + k$, where $k = \min\{f(x) \mid x \in V(G)\}$ and $d_{G-D}(x) \geq f(x)$ for all $x \in V(G)$, then $G - D$ has an f -factor.*

Not only vertex connectivity can be linked with the existence of (g, f) -factors with special properties in bipartite (mg, mf) -graphs. Below, we include some sufficient conditions using edge connectivity for the existence of (g, f) -factors with inclusion and exclusion properties in bipartite (mg, mf) -graphs.

We start with a criterion of f -factors with inclusion and exclusion properties.

Theorem 3.4.18 (Liu and Zang [366]). *Let $G = (X, Y)$ be a bipartite graph and f be an integer-valued function defined on $V(G)$ with $f(X) = f(Y)$. Let C and D be two disjoint subsets of $E(G)$, S and T be any subsets of X and Y , respectively, and $\alpha_C = |C \cap E_G(S, Y - T)|$, $\beta_D = |D \cap E_G(T, X - S)|$. Then G has an f -factor containing C and excluding D if and only if*

$$\gamma_G(S, T; f) = f(S) - f(T) + d_{G-S}(T) \geq \alpha_C + \beta_D.$$

To prove Theorem 3.4.19, we need the following lemma.

Lemma 3.4.3 (Liu and Zang [366]). *Let $G = (X, Y)$ be a bipartite (mf) -graph. Then for any subsets $S \subseteq X$, $T \subseteq Y$, we have*

$$\gamma_G(S, T; f) = \frac{m-1}{m}d_{G-S}(T) + \frac{1}{m}d_{G-T}(S).$$

Theorem 3.4.19 (Liu and Zang [366]). *Let $G = (X, Y)$ be a bipartite (mf) -graph with edge connectivity $\kappa'(G) \geq n$ and $m \geq 2$. Then G has an f -factor containing any given edge and excluding any given $\lfloor \frac{n-1}{2} \rfloor$ edges.*

Proof. We only show the case that n is odd, the other case can be justified likewise. Since G is an (mf) -graph, we have $f(X) = f(Y)$. Let $e_0, e_1, e_2, \dots, e_{\frac{n-1}{2}}$ be any $\frac{n+1}{2}$ edges of G , and $C = \{e_0\}$, $D = \{e_1, e_2, \dots, e_{\frac{n-1}{2}}\}$. We show that G has an f -factor containing C and excluding D . By Theorem 3.4.18, it suffices to show that for any $S \subseteq X$ and $T \subseteq Y$, we have $\gamma_G(S, T; f) = f(S) - f(T) + d_{G-S}(T) \geq \alpha_C + \beta_D$.

By the definitions of α_C and β_D , we clearly have $\alpha_C \leq 1$, $\beta_D \leq \frac{n-1}{2}$, $\alpha_C + \beta_D \leq \frac{n+1}{2}$. Let $k = \min\{f(x) \mid x \in V(G)\}$. Then $n \leq \kappa'(G) \leq \delta(G) \leq mk$. Clearly, $\gamma_G(\emptyset, \emptyset; f) = 0 = \alpha_C + \beta_D$. If $S = \emptyset$ and $T \neq \emptyset$, then $\alpha_C = 0$, $\gamma_G(\emptyset, T; f) = d_G(T) - f(T) = (m-1)f(T) \geq (m-1)k \geq \frac{mk}{2} \geq \frac{n}{2} > \alpha_C + \beta_D$. If $S = X$, then $\beta_D = 0$ and $\gamma_G(X, T; f) = f(X) - f(T) = f(Y) - f(T) = f(Y - T) \geq \alpha_C + \beta_D$. Now we assume that $S \neq \emptyset$ and $S \neq X$. When $T = \emptyset$, we have $\beta_D = 0$ and $\gamma_G(S, \emptyset; f) = f(S) \geq \alpha_C + \beta_D$. When $T = Y$, $\gamma_G(S, Y; f) = f(S) - f(Y) + d_{G-S}(Y) = f(S) - f(X) + mf(X - S) = (m-1)f(X - S) \geq (m-1)k \geq \alpha_C + \beta_D$.

Now we consider that $\emptyset \neq S \subset X$ and $\emptyset \neq T \subset Y$. Since $\kappa'(G) \geq n$, we have $d_{G-S}(T) + d_{G-T}(S) \geq n$. Note that $d_{G-S}(T) \geq \beta_D$, by Lemma 3.4.3, then

$$\begin{aligned} \gamma_G(S, T; f) &= \frac{m-1}{m}d_{G-S}(T) + \frac{1}{m}d_{G-T}(S) \\ &\geq \frac{m-1}{m}d_{G-S}(T) + \frac{1}{m}(n - d_{G-S}(T)) \\ &\geq \frac{m-2}{m}d_{G-S}(T) + \frac{n}{m} \\ &\geq \beta_D - \frac{n-2\beta_D}{m} \\ &\geq \beta_D + \frac{n-n+1}{m}. \end{aligned}$$

Since $\gamma_G(S, T; f)$ is an integer, so $\gamma_G(S, T; f) \geq \beta_D + 1 \geq \alpha_C + \beta_D$. The proof is complete. $\square$

The following example indicates that the edge-connectivity condition $\kappa'(G) \geq n$ in Theorem 3.4.19 is best possible.

Example 3.4.1. Let $G = (X, Y)$ be a bipartite graph. If n is odd, let $X = \{x_i \mid 1 \leq i \leq mk\} \cup \{x'_i \mid 1 \leq i \leq mk\}$, $Y = \{y_i \mid 1 \leq i \leq mk\} \cup \{y'_i \mid 1 \leq i \leq mk\}$, $E(G) = (\{x_i y_j, x'_i y'_j \mid 1 \leq i, j \leq mk\} - \{x_i y_i, x'_i y'_i \mid 1 \leq i \leq \frac{n-1}{2}\}) \cup \{x_i y'_i, x'_i y_i \mid 1 \leq i \leq \frac{n-1}{2}\}$, where k is a constant and $n \leq mk$. Clearly $\kappa'(G) = n - 1$. Let $f(x) = k$ for all $x \in V(G)$ and let $C = \{x_1 y'_1\}$ and $D = \{x_i y_i \mid 1 \leq i \leq \frac{n-1}{2}\}$. For $S = \{x_i \mid 1 \leq i \leq mk\}$ and $T = \{y_i \mid 1 \leq i \leq mk\}$, we have $\gamma_G(S, T; f) = \frac{m-1}{m} d_{G-S}(T) + \frac{1}{m} d_{G-T}(S) = \frac{n-1}{2} < \alpha_C + \beta_D$. By Theorem 3.4.18, G has no f -factors containing C and excluding D . For the case of n is even, replacing $\frac{n-1}{2}$ with $\frac{n-2}{2}$, we can construct a similar graph G such that G has no f -factors containing C and excluding D .

The following results discuss the existence of (g, f) -factors excluding given edges in bipartite graphs.

Theorem 3.4.20 (Bian [80]). Let $G = (X, Y)$ be a bipartite (mg, mf) -graph with edge connectivity $\kappa'(G)$ such that $mk \geq \kappa'(G) \geq n$ and $m \geq 2$, and g and f are two positive integer-valued functions defined on $V(G)$ such that $k \leq g(x) \leq f(x)$, $f(X) = f(Y)$ and $g(X) = g(Y)$ for all $x \in V(G)$. Then G has a (g, f) -factor excluding any given $\lfloor \frac{n+1}{2} \rfloor$ edges.

Corollary 3.4.4 (Bian [80]). Let $G = (X, Y)$ be a bipartite m -regular graph with edge connectivity $\kappa'(G) \geq n$ and $m \geq 2$. Then G has a perfect matching excluding any given $\lfloor \frac{n+1}{2} \rfloor$ edges.

In this section, we discuss sufficient conditions which use vertex or edge-connectivity to insure the existence of (g, f) -factors with inclusion and/or exclusion properties in bipartite (mg, mf) -graphs. However, for non-bipartite graphs, the similar approaches are not explored yet and more research is needed in this direction.

3.5 Other parameters and existence of factors

There are many results regarding the existence of factors in terms of other parameters, such as degree, independent number and degree sum. In this section, we consider these parameters as sufficient conditions. We only provide the proofs for a few theorems, for the rest, the interested readers can refer to the original papers for the proofs.

Theorem 3.5.1 (Egawa and Enomoto [169]). Let G be a graph of order n with $n > k \geq 1$, kn even and $\delta(G) \geq n + 2k - 2\sqrt{kn}$. Then G has a k -factor.

Theorem 3.5.2 (Kano [279]). Let a, b be integers such that $2 \leq a < b$, and let G be a graph of order n with $n \geq 6a + b$. Set $\lambda = \frac{a-1}{b}$. Suppose that, for any subset $X \subset V(G)$, we have $N_G(X) = V(G)$ if $|X| \geq \lfloor \frac{n}{1+\lambda} \rfloor$; $|N_G(X)| \geq (1+\lambda)|X|$ if $|X| < \lfloor \frac{n}{1+\lambda} \rfloor$. Then G has an $[a, b]$ -factor.

Theorem 3.5.3 (Woodall [589]). *Let $k \geq 2$ be an integer and G a connected graph of order n with kn even. Suppose that*

$$|N(X)| \geq \frac{1}{2k-1}(|X| + (k-1)n - 1) \quad (3.56)$$

for every nonempty independent subset X of $V(G)$, and G has minimum degree

$$\delta(G) \geq \frac{(k-1)(n+2)}{2k-1}.$$

Suppose further that, if $n < 4k - 6$, then

$$\delta(G) > n + 2k - 2\sqrt{kn + 2}.$$

Then G has a k -factor.

To replace (3.56) with a minimum degree condition, Niessen [445] obtained a sharp sufficient condition for the existence of k -factor in terms of the minimum degree and the independent number. Furthermore, Niessen (1995) gave a sufficient condition using one constraint involving only minimum degree $\delta(G)$ and independent number $\alpha(G)$.

Theorem 3.5.4 (Niessen [446]). *Let $k \geq 4$ be an integer and G be a graph of order n with $n \geq k + 1$. If $\delta(G) > \frac{k+2}{4}\alpha(G) + \frac{5k-3}{8} - \frac{2}{k}$, then G has a k -factor.*

The following results reveal the relationship between the degree sum of non-adjacent vertices and the existence of factors.

Theorem 3.5.5 (Iida and Nishimura [255]). *Let k be a positive integer and G a graph of order n with $n \geq 4k - 5$, kn even and $\delta(G) \geq k$. If $d_G(u) + d_G(v) \geq n$ for all pairs of non-adjacent vertices u and v of G , then G has a k -factor.*

Theorem 3.5.6 (Nishimura [455]). *Let $k \geq 3$ be an integer and G a connected graph of order n with $n \geq 4k - 3$, kn even and $\delta(G) \geq k$. If $\max\{d_G(u), d_G(v)\} \geq \frac{n}{2}$ for all pairs of non-adjacent vertices u and v of G , then G has a k -factor.*

Liu and Zhang (2000), and Li and Cai (1998), independently, generalized both Theorems 3.5.5 and 3.5.6, but with slight differences in the conditions.

Theorem 3.5.7 (Liu and Zhang [367]). *Let G be a graph of order n with $n \geq \frac{(a+b)(a+b-2)}{b}$. If $\max\{d_G(u), d_G(v)\} \geq \frac{an}{a+b}$ for all pairs of non-adjacent vertices u and v of G , then G has an $[a, b]$ -factor.*

Theorem 3.5.8 (Li and Cai [346]). *Let G be a graph of order n and let a and b be integers with $1 \leq a < b$. If $\delta(G) \geq a$, $n \geq 2a + b + \frac{a^2-a}{b}$, and $\max\{d_G(u), d_G(v)\} \geq \frac{an}{a+b}$ for all pairs of non-adjacent vertices u and v of G , then G has an $[a, b]$ -factor.*

Proof. Suppose, to the contrary, that G has no $[a, b]$ -factors, by Theorem 2.1.3, there exist two vertex disjoint subsets S and T of $V(G)$, where $|S| = s$ and $|T| = t$, such that

$$\Delta = bs - at + \sum_{x \in T} d_{G-S}(x) + 1 \leq 0. \quad (3.57)$$

We choose subsets S and T subject to (3.57) and $|T|$ is minimum. Then it is easy to see that $T \neq \emptyset$ and $e(v, V(G) - S) \leq a - 1$ for all $v \in T$.

Define

$$h_1 = \min\{d_{G-S}(x) \mid x \in T\}.$$

And let x_1 be a vertex in T satisfying $d_{G-S}(x_1) = h_1$. Further, if $T - N_T[x_1] \neq \emptyset$, we define

$$h_2 = \min\{d_{G-S}(x) \mid x \in T - N_T[x_1]\},$$

and let x_2 be a vertex in $T - N_T[x_1]$ satisfying $d_{G-S}(x_2) = h_2$ (see Fig. 3.5). Then we have $h_1 \leq h_2 \leq a - 1$ and $d_G(x_i) \leq s + h_i$ for $i = 1, 2$.

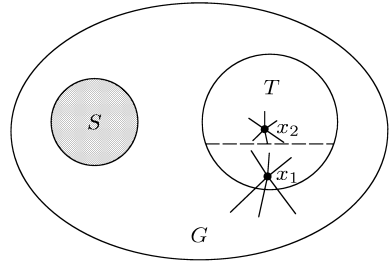


Fig. 3.5 Illustration of the proof.

Furthermore, we have that if $T - N_T[x_1] \neq \emptyset$, then $s + h_2 \geq \frac{an}{a+b}$. Otherwise, we have $s + h_1 \leq s + h_2 < \frac{an}{a+b}$, and this implies $d_G(x_1) < \frac{an}{a+b}$ and $d_G(x_2) < \frac{an}{a+b}$. Because x_1 and x_2 are two nonadjacent vertices in T , that would contradict to the assumption of theorem.

We consider two cases.

Case 1. $T = N_T[x_1]$.

By (3.57) and the definition of h_1 , we have

$$\begin{aligned} 0 &\geq bs - at + \sum_{v \in T} d_{G-S}(v) + 1 \\ &\geq bs - at + h_1 t + 1 \\ &= bs + (h_1 - a)t + 1 \\ &\geq b(a - h_1) + (h_1 - a)t + 1 \quad (\text{since } s + h_1 \geq d_G(x_1) \geq a) \\ &= (a - h_1)(b - t) + 1 \\ &> 0, \quad (\text{since } h_1 \leq a - 1 \text{ and } T = N_T(x_1), t \leq h_1 + 1 \leq a < b) \end{aligned}$$

a contradiction.

Case 2. $T - N_T[x_1] \neq \emptyset$.

Let w be the number of components of $G - (S \cup T)$. Then $w \leq n - s - t$. Let $p = |N_T[x_1]|$, then $h_1 \geq p - 1$. By $T - N_T[x_1] \neq \emptyset$, we have $t \geq p + 1$.

Since $0 \leq w \leq n - s - t$, $a - h_2 \geq 1$, from (3.57), we have

$$\begin{aligned} (n - s - t)(a - h_2) &\geq bs - at + \sum_{v \in T} d_{G-S}(v) + 1 \\ &\geq bs - at + h_1 p + h_2(t - p) + 1 \\ &= bs + (h_1 - a)p + (h_2 - a)(t - p) + 1, \end{aligned}$$

that is,

$$(a - h_2)(n - s) - bs \geq (h_1 - h_2)(h_1 + 1) + 1. \quad (3.58)$$

Because $s + h_2 \geq d_G(x_2) \geq \frac{an}{a+b}$, we have $s - \frac{an}{a+b} \geq -h_2$. By $a + b - h_2 > a - h_2 \geq 1$, it yields

$$\left(s - \frac{an}{a+b}\right)(a + b - h_2) \geq -h_2(a + b - h_2). \quad (3.59)$$

From the assumption, we see $n \geq 2a + b + \frac{a^2 - a}{b}$ and thus $\frac{bn}{a+b} > a + b - 1$. Hence, if $h_2 \geq 0$, then

$$h_2 \left(\frac{bn}{a+b}\right) \geq h_2(a + b - 1); \quad (3.60)$$

and if $h_2 > 0$, then we have

$$h_2 \left(\frac{bn}{a+b}\right) > h_2(a + b - 1). \quad (3.61)$$

Adding (3.58), (3.59) and (3.60), we have

$$\begin{aligned} 0 &\geq (h_1 - h_2)(h_1 + 1) + 1 - h_2(a + b - h_2) + h_2(a + b - 1) \\ &= h_1^2 - (h_2 - 1)h_1 + h_2^2 - 2h_2 + 1 \\ &\geq -\left(\frac{1}{4}\right)(h_2 - 1)^2 + h_2^2 - 2h_2 + 1 \\ &\quad \left(\text{the minimum is reached only if } h_1 = \frac{h_2 - 1}{2}\right) \\ &= \frac{3}{4}(h_2 - 1)^2 \\ &\geq 0. \quad (\text{the minimum 0 is reached only when } h_2 = 1) \end{aligned}$$

The above inequalities imply a contradiction except when $h_2 = 1$. But, for the case of $h_2 = 1$, we have (3.61). So by adding (3.58), (3.59) and (3.61), and using the same arguments, we have $0 > (h_1 - h_2)(h_1 + 1) + 1 - h_2(a + b - h_2) + h_2(a + b - 1) \geq \frac{3}{4}(h_2 - 1)^2 \geq 0$, a contradiction again. The theorem is proved. $\square$

Let G be a complete graph, define the *neighborhood union* of G to be ∞ ; otherwise, define the neighborhood union by

$$NC(G) = \min\{|N_G(u) \cup N_G(v)| \mid u, v \in V(G), uv \notin E(G)\}.$$

In recent years, many researchers consider the relationship between the neighborhood union condition and factors. In [256], Iida and Nishimura gave a neighborhood union condition for the existence of k -factors.

Theorem 3.5.9 (Iida and Nishimura [256]). *Let $k \geq 2$ be a integer and G a connected graph of order n with $n \geq 9k - 1 - 4\sqrt{2}(k - 1)^2 + 2$, and $kn \equiv 0 \pmod{2}$. If $\delta(G) \geq k$ and $|N_G(x) \cup N_G(y)| \geq \frac{n+k-2}{2}$ for all pairs of non-adjacent vertices x and y of G , then G has a k -factor.*

Note that Niessen [447] obtained a more general result using neighborhood union condition. He proved that if $\delta(G) \geq k \geq 2$, $n \geq 8k - 7$ and $|N_G(x) \cup N_G(y)| \geq n/2$, then G has a k -factor or G belongs to two families of exceptional graphs, which have no k -factors ($k \geq 3$).

Matsuda [422] obtained a neighborhood union condition for the existence of $[a, b]$ -factors. We present the result and its proof below. The technique used in the proof is along the same line as in that of Theorem 3.5.8.

Theorem 3.5.10 (Matsuda [422]). *Let a and b be integers with $1 \leq a < b$, and let G be a graph of order n with $n \geq \frac{2(a+b)(a+b-1)}{b}$ and $\delta(G) \geq a$. If $|N_G(x) \cup N_G(y)| \geq \frac{an}{a+b}$ for any pairs of non-adjacent vertices x and y of G , then G has an $[a, b]$ -factor.*

Proof. By contradiction. Assume that G has no $[a, b]$ -factors, by Theorem 2.1.3, there exist two vertex disjoint subsets S and T of $V(G)$ such that

$$b|S| + \sum_{x \in T} (d_{G-S}(x) - a) \leq -1. \quad (3.62)$$

We choose such subsets S and T so that T is minimal.

If $T = \emptyset$, then by (3.62), $b|S| \leq -1$, a contradiction. Thus $T \neq \emptyset$. If there exists a vertex $x \in T$ such that $d_{G-S}(x) \geq a$, then S and $T - \{x\}$ satisfy (3.62), which contradicts with the choice of T . Thus for all vertices $x \in T$, we have $d_{G-S}(x) \leq a - 1$. Define $h_1 = \min\{d_{G-S}(x) \mid x \in T\}$, and say, x_1 be a vertex such that $d_{G-S}(x_1) = h_1$. Note that $h_1 \leq a - 1$ holds. We consider two cases.

Case 1. $T = N_T[x_1]$.

Since $|S| + h_1 \geq d_G(x_1) \geq \delta(G) \geq a$, $h_1 \leq a - 1$ and $|T| = |N_T[x_1]| \leq d_{G-S}(x_1) + 1 = h_1 + 1 \leq a < b$, we have

$$-1 \geq b|S| + \sum_{x \in T} (d_{G-S}(x) - a) \geq b(a - h_1) + (h_1 - a)|T| = (a - h_1)(b - |T|) > 0,$$

a contradiction.

Case 2. $T \neq N_T[x_1]$.

Clearly, $T - N_T[x_1] \neq \emptyset$. Define $h_2 = \min\{d_{G-S}(x) \mid x \in T - N_T[x_1]\}$ and assume a vertex $x_2 \in T - N_T[x_1]$ such that $d_{G-S}(x_2) = h_2$. Note that $h_1 \leq h_2 \leq a - 1$. Since two vertices x_1 and x_2 are non-adjacent, by the neighborhood union condition, we have

$$\frac{an}{a+b} \leq |N_G(x_1) \cup N_G(x_2)| \leq |S| + h_1 + h_2. \quad (3.63)$$

Since $n - |S| - |T| \geq 0$ and $a - h_2 \geq 1$, $(n - |S| - |T|)(a - h_2) \geq 0$. Combining this inequality together with (3.62), then

$$\begin{aligned} (n - |S| - |T|)(a - h_2) &\geq b|S| + \sum_{x \in T} (d_{G-S}(x) - a) + 1 \\ &\geq b|S| + h_1|N_T[x_1]| + h_2(|T| - |N_T[x_1]|) - a|T| + 1 \\ &= b|S| + (h_1 - h_2)|N_T[x_1]| + (h_2 - a)|T| + 1 \\ &\geq b|S| + (h_1 - h_2)(h_1 + 1) + (h_2 - a)|T| + 1, \end{aligned}$$

where $|T| \geq |N_T[x_1]| + 1$, $h_1 - h_2 \leq 0$ and $h_1 + 1 \geq |N_T[x_1]|$. From the inequality above, we have

$$0 \leq (a - h_2)n - (a + b - h_2)|S| + (h_2 - h_1)(h_1 + 1) - 1. \quad (3.64)$$

Using (3.63) and (3.64), $h_1 \leq h_2 \leq a - 1$, $n \geq \frac{2(a+b)(a+b-1)}{b}$, we have

$$\begin{aligned} 0 &\leq (a - h_2)n - (a + b - h_2) \left(\frac{an}{a+b} - (h_1 + h_2) \right) + (h_2 - h_1)(h_1 + 1) - 1 \\ &= -\frac{bn}{a+b}h_2 - h_2^2 + (a + b + 1)h_2 - h_1^2 + (a + b - 1)h_1 - 1 \\ &\leq -\frac{bn}{a+b}h_2 - h_2^2 + (a + b + 1)h_2 - h_2^2 + (a + b - 1)h_2 - 1 \\ &= -2h_2^2 + 2(a + b)h_2 - \frac{bn}{a+b}h_2 - 1 \\ &\leq -2h_2^2 + 2h_2 - 1 \leq -1, \end{aligned}$$

a contradiction. Consequently, the theorem is proved. $\square$

The following example indicates that the neighborhood union condition in Theorem 3.5.10 is best possible, i.e., we can not replace $\frac{an}{a+b}$ by $\frac{an}{a+b} - 1$.

Example 3.5.1. Let $G = (A, B)$ be a complete bipartite graph, where $|A| = at$, $|B| = bt + 1$, and t is any positive integer. Then for any pairs of non-adjacent vertices x and y of B , we have $\frac{an}{a+b} > |N_G(x) \cup N_G(y)| = at > \frac{an}{a+b} - 1$. It is easy to see that G has no $[a, b]$ -factors because $b|A| < a|B|$.

The following result generalizes Theorems 3.5.9 and 3.5.10 and gives the neighborhood union condition for the existence of (g, f) -factors in graphs.

Theorem 3.5.11 (Bian [80]). Let a and b be nonnegative integers and G be a graph of order n with $n > \frac{(a+b)(2a+2b-3)}{a}$ and $\delta(G) \geq \frac{(a+b)b}{2a}$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) < f(x) \leq b$. If $|N_G(x) \cup N_G(y)| \geq \frac{bn}{a+b}$ for any two non-adjacent vertices x and y of G , then G has a (g, f) -factor.

In Section 1.3, we presented several results about the existence of 1-factors in claw-free graphs. Now, we extend the discussion about claw-free graphs to more general factors.

Choudum and Paulraj [148] showed that a connected claw-free graph G with $k|G|$ even and $\delta(G) \geq 2k$ has a k -factor. Egawa and Ota (1991) has weakened the minimum degree condition and obtained the following.

Theorem 3.5.12 (Egawa and Ota [177]). Let G be a connected claw-free graph and k a positive integer with $k|V(G)|$ even. If $\delta(G) \geq \frac{9k+12}{8}$ ($k \geq 2$), then G has a k -factor.

Egawa and Ota also obtained an analogous result for $K_{1,n}$ -free graphs in the same article. Although this result does not imply Theorem 3.5.12, but both proofs are very similar. Moreover, the minimum degree condition in this result may not be sharp, however there are examples to suggest that the minimum degree condition is of the correct order of magnitude.

Theorem 3.5.13 (Egawa and Ota [177]). Let G be a connected $K_{1,n}$ -free graph with $n \geq 3$, and let k be a positive integer and $k \geq n - 1$ when k is odd, $k \geq 2$ when k is even. If $\delta(G) \geq \frac{n^2}{4n-4}k + \frac{3n-6}{2} + \frac{n-1}{4k}$, then G has a k -factor.

Li [342] obtained an analogous form of the above result for $[a, b]$ -factors and Bian (2005) further extended it to the following theorem for (g, f) -factors.

Theorem 3.5.14 (Bian [80]). Let a, b be nonnegative integers and G a connected $K_{1,n}$ -free graph. Let g, f be two integer-valued functions defined on $V(G)$ and for all $x \in V(G)$, $\sum_{x \in V(G)} f(x)$ is even when $g(x) = f(x)$. If $2 \leq n - 1 \leq a \leq g(x) \leq f(x) \leq b$ and $\delta(G) \geq \frac{b}{2} + \frac{(n-1)b^2}{4a} + \frac{a}{4(n-1)} + \frac{(n-1)b}{2a} + \frac{n-1}{4a} + \frac{2n-5}{2}$, then G has a (g, f) -factor.

Cai and Liu [100] studied f -factors in $K_{1,n}$ -free graphs by using the minimum degree as well as independent number as sufficient conditions.

Theorem 3.5.15 (Cai and Liu [100]). *Let G be a connected $K_{1,n}$ -free graph, and let f be a nonnegative integer-valued function defined on $V(G)$ and $1 \leq n-1 \leq a \leq f(x) \leq b$ for all $x \in V(G)$. If $\sum_{x \in V(G)} f(x)$ is even, $\delta(G) \geq b+n-1$ and $\alpha(G) \leq \frac{4a(\delta-b-n+1)}{(b+1)^2(n-1)}$, then G has an f -factor.*

Niessen and Randerath [449] investigated k -factors in regular graphs and showed that every regular graph of *small* order has a k -factor.

Furthermore, reader can find more results about parameters and factors or factorization of graphs in the survey papers [12], [73] and [493].

Chapter 4

Component Factors and Connected Factors

Graph factor problems have been studied extensively for many years. Most of the results mentioned in the previous chapters are of type “degree constraint factors”, i.e., finding a spanning subgraph with given degree conditions. Many of these “degree constraint factors” problems can be transformed to matching problems, and therefore there exist efficient algorithms to determine it. However, there is another type of factor problem, generally referred as “component factor problem”, which consider a spanning subgraph with components isomorphic to the members of a given list. In this chapter, we discuss the component factors and connected factors. There exist characterizations for a few of these problems, but for most cases characterizations have not been discovered yet.

For a set $\{A_1, A_2, A_3, \dots\}$ of graphs, an $\{A_1, A_2, A_3, \dots\}$ -factor of a graph G is a spanning subgraph of G such that each of its components is isomorphic to one of $\{A_1, A_2, A_3, \dots\}$.

One of main research themes in the theory of graph factor is to find a “good” characterization, such as Hall’s condition, Tutte’s condition (1.4) in Theorem 1.3.1. However, very few of such characterizations are known for the component factor problems. Are we unlucky to discovery it or such a characterization may not exist? There is a strong belief among researchers that if a problem is NP-complete, then it has no “good” characterizations. Kirkpatrick and Hell (1983) provided the classifications for complexities of several factor recognition problems.

Denote $\text{FACT}(H)$ the existence problem for H -factors as a language recognition problem:

INSTANCE: A graph G .

QUESTION: Does G admit an H -factor?

Theorem 4.0.1 (Kirkpatrick and Hell [317]).

- (a) Let $\mathfrak{F} = \{H\}$. If $H = K_1$ or $H = K_2$, then $\text{FACT}(\mathfrak{F})$ is in P, otherwise $\text{FACT}(\mathfrak{F})$ is NP-complete;

- (b) Let $\mathfrak{C}$ be any finite set of factor-critical graphs and $\mathfrak{F} = \{K_2\} \cup \mathfrak{C}$. Then $\text{FACT}(\mathfrak{F})$ is in P;
- (c) Let $\mathfrak{F} \subset \{K_1, K_2, K_3, \dots\}$. If $K_1 \in \mathfrak{F}$ or $K_2 \in \mathfrak{F}$, then $\text{FACT}(\mathfrak{F})$ is in P, otherwise $\text{FACT}(\mathfrak{F})$ is NP-complete;
- (d) Let $\mathfrak{F} \subset \{K_{1,1}, K_{1,2}, K_{1,3}, \dots\}$. If $K_{1,t} \notin \mathfrak{F}$ and $K_{1,t+1} \in \mathfrak{F}$ for some $t \geq 1$, then $\text{FACT}(\mathfrak{F})$ is NP-complete, otherwise $\text{FACT}(\mathfrak{F})$ is in P.

As examples, we list the classifications of some factor problems involving cycles and matchings according to the complexity.

- (1) Let H be a 2-connected graph with n vertices and let m be the number of vertices covered by a maximum matching of H . Let $\mathfrak{F} = \{K_2, H\}$. If $m = n$ or $m = n - 1$ and H is factor-critical, then $\text{FACT}(\mathfrak{F})$ is in P. If $m \leq n - 3$, then $\text{FACT}(\mathfrak{F})$ is NP-complete;
- (2) If $\mathfrak{F}$ is a finite family of cycles, then $\text{FACT}(\mathfrak{F})$ is NP-complete;
- (3) For the families of infinite cycles, let $\mathfrak{F} \subseteq \{C_3, C_4, C_5, \dots\}$, then $\text{FACT}(\mathfrak{F})$ is NP-complete except the following cases
 - (a) $\mathfrak{F} = \{C_3, C_4, C_5, \dots\}$;
 - (b) $\mathfrak{F} = \{C_4, C_5, C_6, \dots\}$;
 - (c) $\mathfrak{F} = \{C_5, C_6, C_7, \dots\}$;
 - (d) $\mathfrak{F} = \{C_3, C_5, C_6, C_7, \dots\}$;
- (4) $\text{FACT}(\mathfrak{F})$ is in P for the following cases
 - (a) $\mathfrak{F} = \{K_2, C_3, C_4, \dots\}$;
 - (b) $\mathfrak{F} = \{K_2, C_4, C_5, \dots\}$;
 - (c) $\mathfrak{F} = \{K_2, C_{i_1}, C_{i_2}, \dots, C_{i_t}\}$.

It is no surprise that to determine if a graph has a *connected* f -factor is NP-complete (consider Hamiltonian cycle problem, for instance). In light of complexity of graph factor problems, if the sufficient and necessary conditions may not exist, it is natural to consider the sufficient condition or the relationships between factor problems and graphic parameters, such as degree condition, independent number and others.

4.1 Star factor

An $\{A_1, A_2, A_3, \dots\}$ -factor is called a *star factor* if each A_i is isomorphic to a star K_{1,n_i} , where $i = 1, 2, \dots$ and n_i is a positive integer. In particular, a 1-factor is precisely a $\{K_{1,1}\}$ -factor (or simply, $K_{1,1}$ -factor).

As an example, consider the graphs in Fig. 4.1. The first graph contains a $\{K_{1,1}, K_{1,2}, K_{1,3}\}$ -factor (see Fig. 4.1 (a)), and the second one does not contain $\{K_{1,1}, K_{1,2}, K_{1,3}\}$ -factors (see Fig. 4.1 (b)).

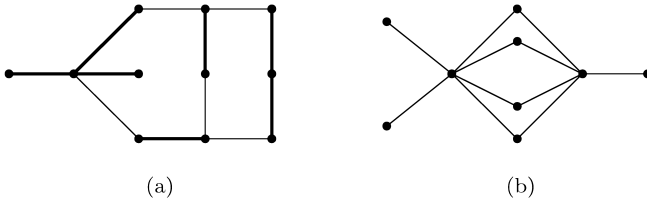


Fig. 4.1 (a) A $\{K_{1,1}, K_{1,2}, K_{1,3}\}$ -factor. (b) No $\{K_{1,1}, K_{1,2}, K_{1,3}\}$ -factor.

Star factors as component factors are the natural one to start with due to their simple structure. The first result of star factors was given by Akiyama, Avis and Era [9]. They proved that a graph G has a $\{K_{1,1}, K_{1,2}\}$ -factor if and only if $i(G - S) \leq 2|S|$ for every $S \subset V(G)$.

For convenience, denote $S_n = \{K_{1,1}, K_{1,2}, \dots, K_{1,n}\}$ and $S_\infty = \{K_{1,1}, K_{1,2}, \dots\}$. It is not hard to see that every connected graph has an S_∞ -factor. In fact, removing a star (both edges and vertices) from a graph recursively while maintaining the connectivity of the remaining subgraph will result a set of vertex-disjoint stars. However, if one limits the size of stars used, then the existence of a star factor is a non-trivial problem. The criterion for limited star factor (i.e., an S_n -factor) was obtained by Amahashi and Kano [31] and Las Vergnas [334], independently.

One may observe that every connected graph with $\Delta(G) \leq n$ has an S_n -factor. Obviously, an S_n -factor is a $[1, n]$ -factor too. On the other hand, if G has a $[1, n]$ -factor F , then each component of F is a subgraph with maximum degree no more than n and thus contains an S_n -factor. Thus G has an S_n -factor. This means that a graph G has an S_n -factor if and only if G has a $[1, n]$ -factor.

There are more equivalent conditions for S_n -factors. Before presenting it, we recall a few notions: $i(G)$ denotes the number of isolated vertices of G . Let H be a subgraph of G , and P is a path of G . Then P is an H -alternating path of G if the edges of P are alternately in $E(H)$ and not in $E(H)$. For a vertex u of G , let $A(u)$ denote the set of vertices w of G such that there exists an H -alternating path from u to w . Furthermore, we define two subsets $OA(u)$ and $EA(u)$ of $A(u)$ to be the sets of vertices w of G such that there exists an H -alternating path from u to w of odd length and of even length, respectively.

Theorem 4.1.1 (Amahashi and Kano [31]; Las Vergnas [334]). *Let G be a graph and $n \geq 2$ be an integer. Then the following statements are equivalent:*

- (a) G has an S_n -factor;
- (b) $i(G - S) \leq n|S|$ for any subset $S \subseteq V(G)$;
- (c) $|N_G(U)| \geq (1/n)|U|$ for any independent set $U \subseteq V(G)$.

Proof. (a) $\Rightarrow$ (b). Suppose that G contains an S_n -factor F . Without loss of generality, let $C_1, C_2, \dots, C_r$ be the components of F . Then, for any vertex subset S of G , we have

$$i(G - S) \leq i(F - S) = \sum_{j=1}^r i(C_j - (S \cap V(C_j))) \leq \sum_{j=1}^r n|S \cap V(C_j)| = n|S|.$$

(b) $\Rightarrow$ (a). Suppose that G has no S_n -factors. Let H be a maximum S_n -subgraph of G and u be a vertex of G not contained in H . Then the following propositions are true.

Claim 1. If $ux_1y_1x_2y_2 \dots x_r y_r$ is an H -alternating path, then $d_H(x_i) = n$ and $d_H(y_i) = 1$ for every i with $1 \leq i \leq r$.

If $d_H(x_1) = d_H(y_1) = 1$, then $H_1 = H \cup ux_1$ is an S_n -subgraph with $V(H_1) = V(H) \cup \{u\}$, a contradiction. If $d_H(x_1) = 1$ and $d_H(y_1) \geq 2$, then $H_2 = H \cup ux_1 - x_1y_1$ is an S_n -subgraph with $V(H_2) = V(H) \cup \{u\}$, a contradiction. If $2 \leq d_H(x_1) \leq n - 1$, then $d_H(y_1) = 1$ and $H_3 = H \cup ux_1$ is an S_n -subgraph with $V(H_3) = V(H) \cup \{u\}$, a contradiction. Hence we have $d_H(x_1) = n$ and $d_H(y_1) = 1$. Similarly, we can show that $d_H(x_2) = n, d_H(y_2) = 1, \dots, d_H(x_r) = n, d_H(y_r) = 1$. For example, if $2 \leq d_H(x_2) \leq n - 1$, then we obtain a contradiction by considering $H \cup y_1x_2 - x_1y_1 \cup ux_1$.

Claim 2. $A(u) - \{u\} \subseteq V(H)$, and $A(u)$ is a disjoint union of $\{u\}, OA(u)$ and $EA(u)$.

It is immediate that $A(u) - \{u\} \subseteq V(H)$. Then, by Claim 1, we have

$$OA(u) = \{w \in A(u) \mid d_H(w) = n\}$$

and

$$EA(u) = \{w \in A(u) \mid d_H(w) = 1\}.$$

Hence, $OA(u) \cap EA(u) = \emptyset$ as $n \geq 2$.

Claim 3. $|EA(u)| = n|OA(u)|$.

For any vertex $w \in OA(u)$, there exist exactly n vertices in $EA(u)$ which are adjacent in H to w but not to any vertex in $OA(u) - \{w\}$. Thus $|EA(u)| = n|OA(u)|$.

Claim 4. Let w be any vertex of G . If w is adjacent to some vertex of $EA(u)$, then $w \in OA(u)$.

Suppose that a vertex w of G is adjacent to a vertex v in $EA(u)$. Let $ux_1y_1 \dots x_r y_r$ ($y_r = v$) be an H -alternating path of even length from u to v . Then $N_H(v) = \{x_r\}$. If w is not in the path, then $ux_1y_1 \dots x_r y_r w$ is an H -alternating path of odd length and so $w \in OA(u)$. If $w = x_j$ for some j ($1 \leq j \leq r$), then $w \in OA(u)$ as $x_j \in OA(u)$. If $w = y_j$ for some j ($1 \leq j \leq r$), then $ux_1y_1 \dots x_j y_j v$ is an H -alternating path of odd length, and thus $v \in OA(u)$, which is contrary to Claim 2. This completes the proof.

Let $S = OA(u)$. Then $EA(u)$ is an independent set of $G - S$ by the above claims. Since u is an isolated vertex of $G - S$, we have

$$i(G - OA(u)) = |EA(u) \cup \{u\}| = n|OA(u)| + 1 > n|OA(u)|,$$

a contradiction.

(a) $\Rightarrow$ (c). Let F be an S_n -factor of G , and let $C_1, C_2, \dots, C_r$ be the components of F . Then, for any independent vertex subset U of G , we have

$$N_G(U) \geq |N_F(U)| = \sum_{j=1}^r |N_{C_j}(V(C_j) \cap U)| \geq \sum_{j=1}^r \frac{1}{n} |V(C_j) \cap U| = \frac{|U|}{n}.$$

(c) $\Rightarrow$ (a). Suppose that G has no S_n -factors. Let H be a maximum S_n -subgraph of G and u be a vertex of G not contained in H . Then $EA(u) \cup \{u\}$ is an independent vertex subset of G by the above claims, and we have

$$|N(EA(u) \cup \{u\})| = |OA(u)| = \frac{1}{n} |EA(u)| < \frac{1}{n} |EA(u) \cup \{u\}|.$$

□

Note that the above theorem holds only for $n \geq 2$ since the equivalent conditions do not imply 1-factor when $n = 1$. For the case of $n = 1$, the conditions (b) and (c) are the equivalence for a different type of factors which is given in the next section (i.e., Theorem 4.2.5).

From Theorem 4.1.1, we easily deduce the following interesting result.

Corollary 4.1.1. *Let G be a graph and $n \geq 2$. If $\Delta(G)/\delta(G) \leq n$, then G has an S_n -factor.*

Proof. For any set $S \subseteq V(G)$, consider the set of edges between S and the set of isolated vertices, $E(S, I(G - S))$. Then

$$i(G - S)\delta \leq |E(S, I(G - S))| \leq \Delta|S|,$$

or $i(G - S) \leq \frac{\Delta}{\delta}|S| \leq n|S|$. From Theorem 4.1.1, G has an S_n -factor. □

Yu [611; 612; 613] also studied the structure of graphs with a unique star factor and found a lower bound for the number of star factors in a graph.

Egawa, Fujita and Ota (2007) considered $K_{1,3}$ -factors and obtained the following.

Theorem 4.1.2 (Egawa, Fujita and Ota [174]). *Let G be a graph of order $4k$ ($k \geq 1$). If $\delta(G) \geq 2k$, then G has a $K_{1,3}$ -factor, unless G is isomorphic to $K_{2k,2k}$ when k is odd.*

There are several generalizations for star factors. Let us see the first one of such extensions, so-called f -star factors. Let $f : V(G) \rightarrow \mathbb{Z}^+$ be a function. A subgraph F is called an f -star if F is a star $K_{1,t}$ and its center satisfies $1 \leq t \leq f(v)$. An f -star factor is a spanning subgraph each of its components is an f -star. For this type of general star factor problem, we have a nice criterion.

Theorem 4.1.3 (Berge and Las Vergnas [77]). *Let G be a graph and $f : V(G) \rightarrow \mathbb{Z}^+$ such that $f(v) \geq 2$ for all $v \in V(G)$. Then G has an f -star factor if and only if*

$$\sum_{x \in N_G(S)} f(x) \geq |S| \quad \text{for all independent } S \subseteq V(G).$$

The further generalization of the above theorem is given below. Note that it also implies Theorem 4.2.5.

Theorem 4.1.4 (Berge and Las Vergnas [77]). *Let G be a graph and $f : V(G) \rightarrow \mathbb{Z}^+$ such that $f(v) \geq 1$ for all $v \in V(G)$. Then there exists a spanning subgraph H of G such that each component of H is either an f -star, or an odd cycle C with $f(x) = 1$ for each vertex $x \in V(C)$ if and only if*

$$\sum_{x \in N_G(S)} f(x) \geq |S| \quad \text{for all independent } S \subseteq V(G).$$

Another extension of star factors is the induced star factors. An *induced star* of G is an induced subgraph isomorphic to $K_{1,r}$, i.e., $K_{1,r}$ can be obtained from G by deleting some vertices of G . An *induced star factor* IS_n (resp. IS_∞) is a subgraph each of its components is an induced star of S_n (resp. S_∞). The criteria for IS_n -factors and IS_∞ -factors are obtained.

A graph is a *cactus* if it is connected and each of its blocks is complete and a cactus is called an *odd-cactus* if each of its blocks is of odd order. The graph shown in Fig. 4.2 is a cactus, which is also an odd-cactus.

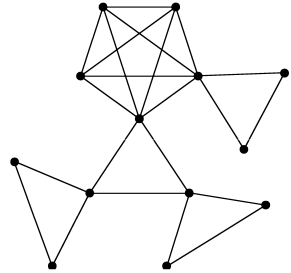


Fig. 4.2 An odd-cactus.

Theorem 4.1.5 (Kelmans [313]; Saito and Watanabe [512]). *A connected graph G has an IS_∞ -factor if and only if G is not an odd-cactus.*

Theorem 4.1.6 (Egawa, Kano and Kelmans [176]). *Let $n \geq 2$ be an integer. A graph G has an IS_n -factor if and only if*

$$oc(G - S) \leq n|S| \quad \text{for all set } S \subseteq V(G),$$

where $oc(G - S)$ denotes the number of those components of $G - S$ that are odd cacti.

The proof is based on the following theorem which is a variant of Hall's Theorem, and can be easily verified by a modification of the known proofs of Hall's Theorem.

Theorem 4.1.7 (Folkman and Fulkerson [212]). *Let $G = (X, Y)$ be a bipartite graph, and $n \geq 1$ be an integer. Then the following are equivalent:*

- (a) $|N_G(S)| \geq |S|$ and $n|N_G(T)| \geq |T|$ for all $S \subseteq X$ and $T \subseteq Y$;
- (b) G has an IS_n -factor in which every vertex of Y is of degree 1.

In order to prove Theorem 4.1.6, we collect some basic facts about cacti in the next lemma.

Lemma 4.1.1 (Egawa, Kano and Kelmans [176]).

- (i) *A cactus with exactly one even block has a 1-factor;*
- (ii) *a cactus with at least one and at most two even blocks has an IS_2 -factor;*
- (iii) *if G is an odd-cactus, then G is factor-critical;*
- (iv) *an odd-cactus does not have an IS -factor.*

Proof. (i) Use induction on $|V(G)|$. Let B be the even block of G . Then B clearly has a 1-factor. Thus if $G = B$, then the proof is complete. Therefore we may assume that $G \neq B$. Clearly every component of $G - V(B)$ is a cactus with exactly one even block. Therefore by the induction hypothesis, they have 1-factors, and hence G also has a 1-factor.

(ii) From (i), we can assume that G has exactly two even blocks (note that G may have odd blocks), say B_1 and B_2 . Clearly there exists a cutvertex x of G such that $B_1 - x$ and $B_2 - x$ are in different components of $G - x$, say in C_1 and C_2 , respectively. If $x \in V(B_i)$ then let x_i be an arbitrary vertex of $B_i - x$ so that $xx_i \in E(G)$. If $x \notin V(B_i)$ then let x_i be an arbitrary vertex of $C_i - B_i$ adjacent to x . Then $\{x, x_1, x_2\}$ induces a 2-star S . Clearly every component of $G - S$ is a cactus having exactly one even block. By (i), every component has a 1-factor, and so G has an IS_2 -factor.

(iii) Since all the components of $G - v$ satisfy the assumptions of (i), it follows that $G - v$ has a 1-factor for every vertex v .

(iv) Use induction on $|V(G)|$. If G consists of exactly one block, then the statement is obvious. So we can assume that G is not a complete graph. Then G has a cutvertex v . For every component C of $G - v$, the subgraph of G induced by $V(C) \cup v$ is an odd-cactus and has no IS -factors, by the induction hypothesis. On the other hand, if G has an IS -factor, then for at least one component D of $G - v$, the subgraph induced by $V(D) \cup v$ must have an IS -factor, a contradiction. $\square$

Proof of Theorem 4.1.6. We first prove the necessity. Suppose that G has a vertex subset S such that $oc(G - S) > n|S|$. Assume, to the contrary, that G has an IS_n -factor F . Since every vertex of S has degree at most n in F , clearly $G - S$ has a component D that is an odd-cactus and that has no common vertices with any star of F having a vertex in S . Therefore $D \cap F$ is an IS_n -factor of D . This contradicts to Lemma 4.1.1 (iv).

We next prove the sufficiency by induction on $|V(G)|$. Obviously we can assume that $|V(G)| \geq 3$. If G is not connected, then each component of G has an IS_n -factor, by the induction hypothesis, and therefore G also has an IS_n -factor. Hence we may assume that G is connected. By taking $S = \emptyset$, it follows that G is not an odd-cactus. Assume that G has two adjacent vertices x and y such that

$$oc(G - S) \leq n|S| - 2n, \text{ for all subsets } S \text{ with } \{x, y\} \subseteq S \subset V(G).$$

Then for every $T \subset V(G - \{x, y\})$,

$$oc(G - \{x, y\} - T) = oc(G - (T \cup \{x, y\})) \leq n|T \cup \{x, y\}| - 2n = n|T|.$$

Hence by the induction hypothesis, $G - \{x, y\}$ has an IS_n -factor, and thus G also has an IS_n -factor. Therefore we may assume that for every pair of adjacent vertices x and y of G , there exists a subset $T_{xy} \subset V(G)$ such that

$$\{x, y\} \subseteq T_{xy} \text{ and } oc(G - T_{xy}) \geq n|T_{xy}| - 2n + 1. \quad (4.1)$$

Let $m = \min\{n|X| - oc(G - X) \mid X \subset V(G) \text{ and } oc(G - X) > 0\}$. Since $|T_{xy}| \geq 2$ and $n \geq 1$, from (4.1), we have $oc(G - T_{xy}) \geq 1$. Therefore m is well-defined. By (4.1) and the assumption, we have $0 \leq m \leq 2n - 1$, and it immediately follows from the definition of m that

$$oc(G - X) \leq n|X| - m$$

for all $X \subset V(G)$ with $oc(G - X) > 0$. A proper subset S of $V(G)$ is said to be *tight* if $n|S| - oc(G - S) = m$ and $oc(G - S) > 0$.

Claim 1. Let S be a tight set of G , and let D be a component of $G - S$ which is not an odd-cactus. Then D has an IS_n -factor.

Indeed, for every $Z \subset V(D)$, we obtain

$$n|S \cup Z| - m \geq oc(G - (S \cup Z)) = oc(G - S) + oc(D - Z) = n|S| - m + oc(D - Z),$$

which means that $oc(D - Z) \leq n|Z|$ because $S \cap Z = \emptyset$. Since $|V(D)| < |V(G)|$, the component D has an IS_n -factor, by the induction hypothesis. Hence Claim 1 is proved.

We consider two cases.

Case 1. $oc(G - S) \geq |S|$ for some tight set S of G .

We define a bipartite graph H with bipartition (S, T) as follows: T is the set of components of $G - S$ that are odd-cacti, and vertices $x \in S$ and $C \in T$ are adjacent in H if and only if x is adjacent to a vertex in the component C of $G - S$ in G .

We first prove that

$$|N_H(X)| \geq |X| \text{ and } n|N_H(Y)| \geq |Y|, \quad (4.2)$$

for all $X \subseteq S$ and $Y \subseteq T$.

If $X = \emptyset$, then clearly $|N_H(X)| \geq |X|$. Suppose that $X \neq \emptyset$ and $|N_H(X)| < |X|$. Then

$$oc(G - (S - X)) \geq oc(G - S) - |N_H(X)| > n|S| - m - |X| \geq n|S - X| - m,$$

because $n \geq 1$. So we have $n|S - X| - oc(G - (S - X)) < m$. This contradicts the choice of m because by $oc(G - S) \geq |S|$, we have $oc(G - (S - X)) \geq oc(G - S) - |N_H(X)| > |S| - |X| \geq 0$. Thus $|N_H(X)| \geq |X|$ for every subset X of S .

Let $Y \subseteq T$. Then by the assumption, $n|N_H(Y)| \geq oc(G - N_H(Y)) \geq |Y|$. Hence the inequality (4.2) holds.

Therefore by Theorem 4.1.7, the graph H has an IS_n -factor F in which every vertex of T is of degree 1. Consequently, by Lemma 4.1.1 and Claim 1, we can obtain an IS_n -factor of G by making use of F .

Case 2. $oc(G - S) < |S|$ for every tight set S of G .

Let S be a tight set of G . We shall first prove that $|S| = 2$ and $m = 2n - 1$. Recall that $m \leq 2n - 1$. If $|S| = 1$, then $oc(G - S) \geq |S|$ as $oc(G - S) > 0$, which contradicts the assumption of this case. So $|S| \geq 2$. If $|S| \geq 3$, then since $n \geq 2$ we have

$$oc(G - S) = n|S| - m \geq n|S| - 2n + 1 = (n - 2)(|S| - 2) + |S| - 3 + |S| \geq |S|,$$

a contradiction. Hence $|S| = 2$. If $m < 2n - 1$, then $oc(G - S) = n|S| - m = 2n - m \geq 2 = |S|$, and so $oc(G - S) \geq |S|$, a contradiction again. Therefore $|S| = 2$ and $m = 2n - 1$.

Let x and y be two adjacent vertices of G . Then by (4.1), there exists a vertex subset T_{xy} of G such that $\{x, y\} \subseteq T_{xy}$ and $m \leq n|T| - oc(G - T_{xy}) \leq 2n - 1 = m$. Therefore T_{xy} is a tight set of G , and so by the above observation,

$T_{xy} = \{x, y\}$ and $oc(G - \{x, y\}) = 1$. In particular, $\{x, y\}$ is a tight set, and if $G - \{x, y\}$ is connected, then $G - \{x, y\}$ is an odd-cactus.

For an edge $a = uv$ of G , let $f(a)$ denote the number of vertices of a largest component of $G - \{u, v\}$. An edge e is called *extremal* if $f(e) = \max\{f(a) \mid a \in E(G)\}$. Let $e = xy$ be an extremal edge of G . We want to prove that $G_1 = G - \{x, y\}$ is connected. Otherwise, let B be a largest component of G_1 . Let R denote the set of edges of G not incident to B and distinct from $e = xy$. Since G_1 is disconnected, the set R is non-empty. Let $a = uv \in R$. Clearly, if there is a vertex $z \in \{x, y\} - \{u, v\}$ that is adjacent to B , then $f(a) > f(e)$, a contradiction. Since a is an arbitrary edge in R , every edge in R is incident to the same end of e , say x , and the other end y is not adjacent to B . Therefore y is an isolated vertex in $G - x$. Since an isolated vertex is an odd cactus by definition, $oc(G - \{x\}) \geq 1 > 0$. Since $n|\{x\}| - m = -n + 1 \leq 0$, we have $oc(G - \{x\}) > n|\{x\}| - m$, which contradicts the definition of m .

Suppose that G_1 has a vertex v which is adjacent to exactly one of x, y . Then $G_1 - v$ has an IS_n -factor, say F , by Lemma 4.1.1 (iii), and $\{x, y, v\}$ induces a 2-star S in G . Thus $F \cup S$ is an IS_n -factor of G . Hence we may assume that $N_G(x) \cap V(G_1) = N_G(y) \cap V(G_1) := N$. If $|N| = 1$ or N is the vertex set of a block of G_1 , then G is an odd-cactus, a contradiction. Therefore there is a matching xx', yy' in G with $\{x', y'\} \subset V(G_1)$. Suppose that x' and y' are not contained in the same block of G_1 . Then every component of $G_2 = G_1 - \{x', y'\}$ is a cactus with at least one and at most two even blocks. By Lemma 4.1.1 (ii), G_2 has an IS_n -factor, say F , and so F together with the matching xx', yy' forms an IS_n -factor of G . Thus we may assume that $N \subseteq V(Q) - v$ for some block Q of G_1 and $v \in V(Q)$. Then $\{x, x', v\}$ induces a 2-star S in G . Obviously, every component of $G_3 = G_1 - \{v, x', y'\}$ is a cactus with exactly one even block. By Lemma 4.1.1 (i), G_3 has an IS_n -factor, say F , and so F together with the star S and the edge yy' forms an IS_n -factor of G . $\square$

4.2 Path and cycle factors

Let P_k ($P_{\geq k}$) denote the path of order k (order at least k) and let C_k be the cycle of order k . If a graph G contains a spanning subgraph H , such that each component of H is isomorphic to some $P_{\geq 2}$ -path, then H is called a *path-factor* of G . If each component of H is isomorphic to P_k , then H is called a *P_k -factor* of G .

Note that a 2-factor is the same as $\{C_i \mid i \geq 3\}$ -factor. We can restate the earlier results of Babler and Petersen (i.e., Corollary 3.4.1) as follows.

Theorem 4.2.1. *Every 2-edge-connected r -regular graph has a $\{C_i \mid i \geq 3\}$ -factor.*

For cubic graphs, the above theorem can be strengthened to the following.

Theorem 4.2.2 (Kawarabayashi *et al.* [308]). *Every 2-connected cubic graph has a $\{C_n \mid n \geq 4\}$ -factor.*

Proof. We use induction on $|V(G)|$. Let G be a 2-connected cubic graph. Then G has a 2-factor F by Theorem 4.2.1. We may assume that F contains a triangle $T = u_1u_2u_3$ as a component. Let G' be the graph obtained from G by contracting T into a single vertex u . It is easy to see that G' is also a 2-connected cubic graph. (Since G is cubic, for each vertex $x \in V(G) - V(T)$, $|N_G(x) \cap V(T)| = 3 - d_{G-T}(x) \leq 3 - d_F(x) = 1$. Thus, G' does not contain a multiple edge.) By the induction hypothesis, G' has a 2-factor F' in which each component is a cycle of length at least four. Suppose that $\{uv_1, uv_2\} \subset E(F')$. Without loss of generality, let $N_G(v_1) \cap V(T) = \{u_1\}$ and $N_G(v_2) \cap V(T) = \{u_2\}$. Then, by replacing the path v_1uv_2 in F' with $v_1u_1u_3u_2v_2$ in G , we obtain a desired 2-factor of G . $\square$

For the bipartite graphs, Theorem 4.2.2 can be strengthened to the following theorem.

Theorem 4.2.3 (Kano, Lee and Suzuki [286]). *Every connected cubic bipartite graph has a $\{C_n \mid n \geq 6\}$ -factor.*

Besides star factors mentioned in the previous section, another natural analogue of matching problem is so-called 2-matching problem (or $\{K_2, C_i \mid i \geq 3\}$ -factor problem, among many other names). This problem has much in common with matching problem and surprisingly, it is much simpler than matching problem in many respects. In fact, this problem is more or less equivalent to bipartite matching problem! It can be considered as a relaxation of matching problem or Hamiltonian cycle problem.

Let us start with a couple of definitions which come from different respective backgrounds but, as we will see it later, they are all closely related.

Let $f : \{0, 1, 2\} \rightarrow E(G)$ be an assignment such that the sum of weights of edges incident with any vertex is at most 2, i.e., $\sum_{e \sim v} f(e) \leq 2$ for any vertex $v \in V(G)$. A 2-matching is a subgraph induced by the edges with weight 1 or 2. The sum of all weights, i.e., $\sum_{e \in E(G)} f(e)$, is called *size* of a 2-matching f . A 2-matching is *perfect* if $\sum_{e \sim v} f(e) = 2$ for every vertex $v \in V(G)$. Clearly, a 2-matching is perfect if and only if its size is $|V(G)|$. A graph G is called *2-matching-critical* if $|V(G)| \geq 2$ and $G - v$ contains a 2-perfect matching for every vertex $v \in V(G)$.

Clearly, the union of any two matchings is a 2-matching, but reverse may not be true. Moreover, each component of a 2-matching is either a path or a cycle. For a perfect 2-matching, each component can only be K_2 or a cycle. Since an even cycle contains a perfect matching, each perfect 2-matching can be transformed to a $\{K_2, C_{2t+1} \mid t \geq 1\}$ -factor, and of course, a $\{K_2, C_{2t+1} \mid t \geq 1\}$ -factor itself is a perfect 2-matching. Therefore, a bipartite

graph has a perfect 2-matching if and only if it has a perfect matching. Now we have the following equivalence:

$$\text{perfect 2-matching} \Leftrightarrow \{K_2, C_i \mid i \geq 3\}\text{-factor} \Leftrightarrow \{K_2, C_{2t+1} \mid t \geq 1\}\text{-factor}.$$

Just like max-min theorem of matchings to covers in bipartite graphs, 2-matchings can be also related to 2-coverings in general graphs. A *2-covering* of a graph G is an assignment of weights 0, 1, and 2 to $V(G)$ such that the sum of weights associated with the two end-vertices of any edge is at least 2. The sum of all weights is called the *size* of the 2-covering. The size of a minimum 2-covering of G is denoted by $\tau_2(G)$. If we assign all vertices with 1's, then it results a 2-covering. Thus $\tau_2(G) \leq |V(G)|$.

Similar to König's max-min theorem regarding maximum matchings and minimum coverings in bipartite graphs, Tutte (1953) proved the following result for 2-matchings and 2-coverings.

Theorem 4.2.4 (Tutte [562]). *For any graph G , then*

$$\max\{\text{all 2-matchings}\} = \min\{\text{all 2-coverings}\}.$$

The following equivalent conditions are also obtained by Tutte (1953).

Theorem 4.2.5 (Tutte [562]). *Let G be a connected graph. Then the following statements are equivalent:*

- (a) G has a perfect 2-matching;
- (b) G has a $\{K_2, C_{2t+1} \mid t \geq 1\}$ -factor;
- (c) $i(G - S) \leq |S|$ for all subsets $S \subset V(G)$;
- (d) $|N(T)| \geq |T|$ for all independent subsets $T \subset V(G)$;
- (e) $|N(S)| \geq |S|$ for all subsets $S \subset V(G)$.

Proof. The equivalences are clear except (a) $\Leftrightarrow$ (d). So we only give the proof of (a) $\Leftrightarrow$ (d).

Suppose that G has a perfect 2-matching M and T is any independent set of $V(G)$. From the discussion above, M contains a subgraph M' in which each component is either K_2 or odd cycle. Hence, an independent set T has at least $|T|$ neighbors in M' . Therefore, $|N(T)| \geq |T|$ for any independent subsets T .

Conversely, for any minimum 2-covering, let T denote the vertices with weight 0, then T must be an independent set and the neighbors of T have weight 2. The remaining vertices have weight 1, otherwise if a vertex has a weight 2, then we can lower its weight to 1 and still is a 2-cover, a contradiction. Therefore, $\tau_2(G) = \min\{|V(G)| - |T| + |N(T)| \mid T \text{ is an independent set}\}$.

Since $|N(T)| \geq |T|$ for any independent subsets T , thus $\tau_2(G) \geq |V(G)|$ and thus $\tau_2(G) = |V(G)| = \max\{\text{all 2-matchings}\}$ from Theorem 4.2.4. Hence, a maximum 2-matching has the size $|V(G)|$, i.e, it is a perfect 2-matching. $\square$

For 2-matching-critical graphs, Pulleyblank (1979) obtained a characterization and some interesting properties for this family of graphs.

Theorem 4.2.6 (Pulleyblank [501]). *A graph G is 2-matching-critical if and only if $|N(S)| > |S|$ holds for all independent set S , where $\emptyset \neq S \subset V(G)$.*

Corollary 4.2.1.

- (a) *Every factor-critical graph is 2-matching-critical;*
- (b) *every 2-matching-critical graph contains a perfect 2-matching.*

The next theorem demonstrates a relation between the matchings and 2-matchings.

Theorem 4.2.7 (Balas [65]). *Let G be a graph. There exists a maximum matching in G which is contained in a maximum 2-matching.*

At this point, we would like to introduce a new concept – regularization. It seems that it is odd time to bring it in, since, just from the definitions of 2-matching and regularization, they are totally two different concepts. However, we will see from the up-coming results that they are actually closely related. A graph G is called *regularizable* if by replacing some edges in G by multiple edges to obtain a regular graph.

Theorem 4.2.8 (Berge (1978)). *Let G be a connected graph. Then the following statements are equivalent:*

- (a) *G is regularizable;*
- (b) *G is either an elementary bipartite graph or a 2-matching-critical graph;*
- (c) *for each edge $e \in E(G)$, there exists a perfect 2-matching of G in which e has weight 1 or 2.*

Proof. Due to the limited space, we only present the proof of equivalence of (a) and (c).

Suppose that G is regularizable, by definition, each edge of G can be replaced by parallel edges to obtain a regular graph G' . Without loss generality, we assume that G' is of even degree. Otherwise, we can double every edge of G' to get such a graph. From Petersen's Theorem, G' can be decomposed into 2-factors. Let $e \in E(G)$ and F be a 2-factor of G' containing any edge parallel to e . For $f \in E(G)$, let $\omega(f)$ denote the number of parallel edges of f in F . Then the function $\omega(f)$ is a perfect 2-matching of G in which e has positive weight.

Conversely, assume that each edge e of G is contained in some perfect 2-matching ω_e . For each $f \in E(G)$, let $\omega(f) = \sum_e \omega_e(f)$. Replace each edge f by $\omega(f)$ parallel edges. By assumption $\omega(f) > 0$ for every edge f , so at least one new edge is substituted for every old edge. The resulting graph is clearly $(2|E(G)|)$ -regular. Thus G is regularizable. $\square$

Let us mention one more result related to 2-matchings, a sufficient condition from Ryjáček.

Theorem 4.2.9 (Ryjáček [507]). *Let G be a connected claw-free graph of odd order. If G has at most one leaf, then G has a perfect 2-matching.*

Before turning our attention to path-factors, we would like to mention another factor problem related to 2-matchings.

We consider $\{K_2, C_i \mid i \geq 4\}$ -factor, which is more restrictive than 2-matching. A *triangle cluster* is a connected graph whose edges can be partitioned into disjoint triangles with the property that any two triangles have at most one vertex in common and if such a vertex exists, then it is a cutvertex of the triangle cluster. We use $tc(G)$ to denote the number of triangle clusters in G .

Theorem 4.2.10 (Cornuéjols and Pulleyblank [155]). *A graph G has a $\{K_2, C_i \mid i \geq 4\}$ -factor if and only if*

$$i(G - S) + tc(G - S) \leq |S|$$

for all $S \subset V(G)$.

Recall that P_k ($P_{\geq k}$) denotes the path of order k (order at least k).

Theorem 4.2.11 (Akiyama and Kano [13]). *Every maximal planar graph has a path-factor, or a $\{P_2, P_3\}$ -factor.*

Kaneko [269] obtained a nice characterization for $P_{\geq 3}$ -factors. This is a deep result and its proof is quite involved, its structure is well understood. Moreover, it has many nice corollaries. To define the “trouble block” for $P_{\geq 3}$ -factors, we need to introduce a family of graphs $\mathcal{A}$ which is defined recursively, and the procedure is very similar to ear-decompositions in Section 5.2. Given a subgraph H , an *open H -ear* (*closed H -ear*) is a path (cycle) of odd length such that only its end-vertices (only one vertex) are in H . The class of graphs, $\mathcal{A}$, is constructed recursively as follows:

- (1) $K_1 \in \mathcal{A}$, and
- (2) let $G \in \mathcal{A}$ and G' is obtained from G by attaching an open (or closed) G -ear to G , then $G' \in \mathcal{A}$.

Using inductive arguments, it is easy to see that each member of $\mathcal{A}$ has odd order and is a factor-critical graph. Note that if $G \in \mathcal{A}$ and $xy \notin E(G)$, where $x \neq y$, then $G \cup xy$ is also in $\mathcal{A}$. In particular, $K_{2m+1} \in \mathcal{A}$.

For any graph G of order n , define $Sun(G)$ to be a graph obtained from G by attaching a leaf to each vertex of G (that is, $Sun(G)$ has $2n$ vertices and n leaves). Then define

$$\mathcal{C} = \{K_1\} \cup \{\text{Sun}(G) \mid G \in \mathcal{A}\}.$$

A graph belonging to $\mathcal{C}$ is called a *sun*.

For a graph $G \in \mathcal{C}$ and $G \neq K_1, K_2$, then G has no $P_{\geq 3}$ -factors. Otherwise, suppose that there exists a $P_{\geq 3}$ -factor H . Let L be the set of end-vertices of path components in H (of course, $|L|$ is even). By the structure of G , then the set of all leaves in G is the same as L and $H - L \in \mathcal{A}$. However, $|H - L| = |L|$ is even and every member of $\mathcal{A}$ is factor-critical, which is impossible.

Let $c_s(G)$ denote the number of sun components of G . Now we present the characterization of $P_{\geq 3}$ -factors.

Theorem 4.2.12 (Kaneko [269]). *A graph G has a $P_{\geq 3}$ -factor if and only if*

$$c_s(G - S) \leq 2|S|,$$

for every subset S of $V(G)$.

Note: Since a $P_{\geq 3}$ -factor is equivalent to a $\{P_3, P_4, P_5\}$ -factor, the above theorem can be restated as a criterion of $\{P_3, P_4, P_5\}$ -factors.

Theorem 4.2.12 has several interesting consequences as shown below. They all can be deduced from Theorem 4.2.12 easily. But here we only give the proof of the first one.

Corollary 4.2.2 ([269]). *Every r -regular graph ($r \geq 2$) has a $P_{\geq 3}$ -factor.*

Proof. Suppose, to the contrary, that G has no $P_{\geq 3}$ -factors. By Theorem 4.2.12, there exists a subset S of $V(G)$ such that $G - S$ has at least $2|S| + 1$ sun components. Let s_1 be the number of singletons of $G - S$ and s_2 be the number of sun components H of $G - S$ such that $H \in \mathcal{C} - \{K_1\}$. Hence $s_1 + s_2 \geq 2|S| + 1$. Since every member of $\mathcal{C} - \{K_1\}$ has leaves and G is r -regular, it follows

$$r|S| \geq rs_1 + (r - 1)s_2 \geq (r - 1)(s_1 + s_2).$$

Therefore, we have $r|S| \geq (r - 1)(2|S| + 1)$ and thus

$$r \leq \frac{2|S| + 1}{|S| + 1} = 2 - \frac{1}{|S| + 1},$$

a contradiction to $r \geq 2$. □

Corollary 4.2.3 ([269]). *If G is a 1-tough graph with at least three vertices, then G has a $P_{\geq 3}$ -factor. Moreover, if $\delta(G) \geq 2$, then a $1/2$ -tough graph has a $P_{\geq 3}$ -factor.*

Corollary 4.2.4 ([269]). *Every 3-connected planar graph has a $P_{\geq 3}$ -factor.*

Corollary 4.2.5 ([269]). *Every claw-free graph with $\delta(G) \geq 2$ has a $P_{\geq 3}$ -factor.*

Corollary 4.2.5 is generalized by Ando *et al.* (2002).

Theorem 4.2.13 (Ando *et al.* [56]). *Let G be a claw-free graph and $\delta(G) \geq d$. Then G has a $P_{\geq d+1}$ -factor.*

If the assumption “2-connected” is added, the path bound “ $d + 1$ ” may not be best possible. They posed the following conjecture.

Conjecture 4.2.1 (Ando *et al.* [56]). *Let G be a 2-connected claw-free graph with $\delta(G) \geq d$. Then G has a $P_{\geq 3d+3}$ -factor.*

There are examples (Fig. 4.3) to show that the conjecture is sharp.

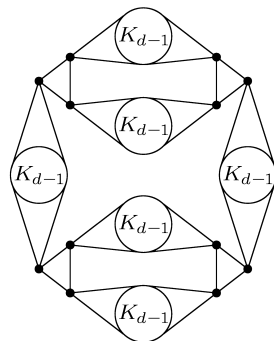


Fig. 4.3 Conjecture 4.2.1 is sharp.

Theorem 4.2.14 (Akiyama and Kano [13]). *Every 3-connected cubic graph of order $4m$ has a P_4 -factor.*

A conjecture related to the above result was posed by Akiyama and Kano (1985) and is still open.

Conjecture 4.2.2 (Akiyama and Kano [12]). *Every 3-connected cubic graph of order divisible by three has a P_3 -factor.*

It was pointed out that the condition of connectivity can not be improved, i.e., there exists a 2-connected cubic planar graph of order $3p$ (Fig. 4.4), which does not contain P_3 -factors [308]. Kawarabayashi *et al.* [308] got the following result which is the best result about Conjecture 4.2.2 at this point.

Theorem 4.2.15 (Kawarabayashi *et al.* [308]). *Every 2-connected cubic graph has a $\{P_{\geq 6}\}$ -factor. In particular, it has a $\{P_3, P_4\}$ -factor.*

In order to prove Theorem 4.2.15, we need the following two lemmas.

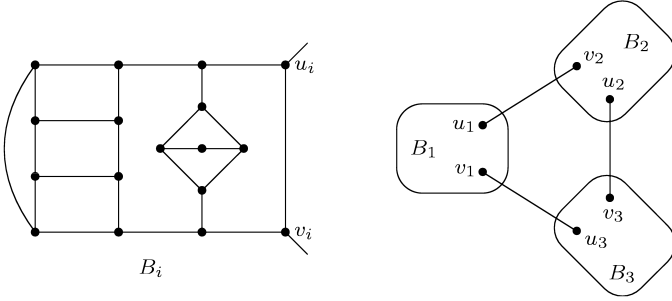


Fig. 4.4 A 2-connected cubic planar graph without $\{P_3\}$ -factors.

Lemma 4.2.1 ([308]). *Let H be a connected multigraph with $\delta(H) \geq 2$. Then H has a spanning subgraph F such that $1 \leq d_F(x) < d_H(x)$ for every vertex $x \in V(H)$, unless H is an odd cycle.*

Proof. Let H be a connected multigraph with $\delta(H) \geq 2$. We shall prove the following slightly stronger statement.

(*) For any given vertex $v \in V(H)$, there exists a spanning subgraph F of H such that $1 \leq d_F(x) \leq \frac{d_H(x)+1}{2}$ for every vertex $x \in V(H) - \{v\}$, and $1 \leq d_F(v) \leq \frac{d_H(v)}{2} + 1$.

First, we show that (*) implies the lemma. If H is a cycle, then the result can be easily checked. Otherwise, take a vertex $v \in V(H)$ with $d_H(v) \geq 3$, and apply (*).

In order to prove (*), we consider an Eulerian supergraph H^* of H obtained by adding independent edges joining the vertices with odd degree. Let

$$v_0 e_1 v_1 e_2 v_2 \dots e_m v_m$$

be an Eulerian circuit of H^* with $v_0 = v_m = v$. If $d_H(v)$ is odd, then we may assume that $e_1 \notin E(H)$. Let F^* be the spanning subgraph induced by the edges with odd indices. Then $d_{F^*}(x) = \frac{d_{H^*}(x)}{2}$ if $x \neq v$, and $d_{F^*}(v) = \frac{d_{H^*}(v)}{2}$ or $\frac{d_{H^*}(v)}{2} + 1$ depending on the parity of $m = |E(H^*)|$. Let F be the spanning subgraph of H induced by $E(F^*) \cap E(H)$. Then by the definition of H^* , we can easily check that $1 \leq d_F(x) \leq \frac{1+d_H(x)}{2}$ for each $x \in V(H) - \{v\}$, and $1 \leq d_F(v) \leq \frac{d_H(v)}{2} + 1$. This proves (*). $\square$

Lemma 4.2.2 ([308]). *Let C be a cycle and $W \subset V(C)$ be a nonempty vertex subset. Then there exists a path factor L of C such that every vertex of W has degree one in L and that every component of L contains one or two vertices of W .*

Proof. At first, give an orientation $\vec{C}$ to C . Let $w_1, w_2, \dots, w_m$ be the vertices of W . Assume that they appear in this order along $\vec{C}$, and that $w_1^- \notin W$, since $W \neq V(C)$.

Choose subpaths of C according to the following algorithm, and define L as the union of these paths.

- (1) Choose $w_1 \overleftarrow{C} w_m^+$. (The edge $w_m w_m^+$ might be chosen later.) Set $i = 2$;
- (2) repeat (3), while $i \leq m$;
- (3) if $w_i^- \neq w_{i-1}$, then choose $w_i \overleftarrow{C} w_{i-1}^+$ and set $i := i + 1$. Otherwise, then $w_i^- = w_{i-1}$. If $i < m$, then choose the path $w_i \overrightarrow{C} w_{i+1}$ and set $i := i + 2$. If $i = m$, then choose the edge $w_m w_m^+$ to be added to the path $w_1 \overleftarrow{C} w_m^+$.

It is easy to see that L satisfies the desired conditions. $\square$

Proof of Theorem 4.2.15. Let G be a 2-connected cubic graph of order at least six. Then by Theorem 4.2.2, G has a 2-factor in which each component is a cycle of length at least four. Let $\mathcal{C}$ denote the set of cycles in the 2-factor. Let H be the multigraph obtained from G by contracting each cycle in $\mathcal{C}$ into a single vertex (and deleting all resulting loops). For each vertex $v \in V(H)$, let $C_v \in \mathcal{C}$ denote the corresponding cycle in G . Note that the edges of H correspond to the independent edges of G , since G is cubic. Since G is 2-connected, H is 2-edge-connected.

Suppose first that H is not an odd cycle. Then by Lemma 4.2.1, there exists a spanning subgraph F such that $1 \leq d_F(v) < d_H(v)$ for every vertex $v \in V(H)$. We assume that F is chosen to be edge-minimal. If H is an odd cycle, we take an arbitrary edge-minimal subgraph F such that $1 \leq d_F(v)$ for every vertex $v \in V(H)$. Then each component of F is a star. Let M be the subgraph of G induced by the edges corresponding to the edges of F . For each cycle $C_v \in \mathcal{C}$ in G , let W_v be the set of vertices in C_v which are incident with an edge in M . Since $1 \leq d_F(v) < \max\{3, d_H(v)\} \leq |V(C_v)|$, we have $\emptyset \neq W_v \subset V(C_v)$. Then by Lemma 4.2.2, there exists a path factor L_v of C_v such that every vertex of W_v has degree one in L_v and that every component of L_v contains one or two vertices of W_v . Define

$$L = \bigcup_{v \in V(H)} L_v \cup M.$$

It is easy to see that each component of L is a path, and contains an edge of M . Take a component of L containing an edge $xy \in E(M)$. Let $C_u, C_v \in \mathcal{C}$ be the cycles such that $x \in V(C_u)$ and $y \in V(C_v)$. Since uv is an edge of a star, we may assume that $d_F(u) = 1$. Then $|W_u| = 1$, and hence L_u has the unique component of order $|L_u| = |V(C_u)| \geq 4$. It is obvious that the component of L_v containing y has at least two vertices. Hence, the order of the component containing xy is at least six, that is, each component of L is a path of order at least six. We complete the proof. $\square$

4.3 El-Zahar's Conjecture and other component factors

For cycle-factors, there is an interesting conjecture posed by El-Zahar (1984).

Conjecture 4.3.1 (El-Zahar [182]). *If G is a graph with $n = n_1 + n_2 + \cdots + n_k$ vertices ($n_i \geq 3$) and $\delta(G) \geq \lceil \frac{1}{2}n_1 \rceil + \lceil \frac{1}{2}n_2 \rceil + \cdots + \lceil \frac{1}{2}n_k \rceil$, then G has a spanning subgraph consisting of cycles of lengths $n_1, n_2, \dots, n_k$.*

Note that the graph $K_{s-1} \vee K_{\lceil \frac{n-s+1}{2} \rceil, \lceil \frac{n-s+1}{2} \rceil}$ has minimum degree $(n+s)/2$ or $(n+s-1)/2$ but contains no s vertices disjoint cycles of odd lengths. Thus, the conjecture is best possible.

El-Zahar [182] provided an affirmative answer to the case $k = 2$, while Dirac's Theorem (i.e., Theorem 4.4.1) is equivalent to the case of $k = 1$.

This conjecture is still wide open, however, some partial results are known. For $n_1 = n_2 = \cdots = n_k = 3$, Corrádi and Hajnal (1963) provided an affirmative answer to El-Zahar conjecture. For $n_1 = n_2 = \cdots = n_k = t \geq 4$, Abassi (1999) proved that El-Zahar's conjecture is true under the condition $\delta(G) \geq \frac{\lceil t/2 \rceil}{t}|V(G)|$ when $|V(G)|$ is sufficiently large. Note that the case of $t = 4$ was settled by Komlós, Sárközy and Szemerédi (2001).

Aigner and Brandt (1993) proved a result stronger than El-Zahar's conjecture but under a stronger condition $\delta(G) \geq (2n-1)/3$. This result was originally conjectured by Sauer and Spencer (1978). Alon and Fischer (1996) also provided a solution to the Sauer-Spencer conjecture (the case of $\delta = 2n/3$) and used work dependent on Regularity Lemma and thus holds only for large graphs.

Theorem 4.3.1 (Aigner and Brandt [7]). *Let G be a graph of order n with $\delta(G) \geq (2n-1)/3$. Suppose $n \geq n_1 + n_2 + \cdots + n_k$, where $n_i \geq 3$ is any integer ($1 \leq i \leq k$). Then G contains the vertex disjoint cycles $C_{n_1}, C_{n_2}, \dots, C_{n_k}$.*

Analogue but weaker version of El-Zahar's conjecture for path-factors under the similar condition was studied by Johansson (1998).

Theorem 4.3.2 (Johansson [266]). *Let G be a connected graph on $n = n_1 + n_2 + \cdots + n_k$ vertices, where $n_i \geq 2$ for all i , $1 \leq i \leq k$, and $\delta(G) \geq \lfloor \frac{1}{2}n_1 \rfloor + \lfloor \frac{1}{2}n_2 \rfloor + \cdots + \lfloor \frac{1}{2}n_k \rfloor$. Then G contains a spanning subgraph consisting of paths of orders $n_1, n_2, \dots, n_k$.*

Let $\lambda = \lambda(n_1, n_2, \dots, n_k)$ denote the number of even integers in the set $\{n_1, n_2, \dots, n_k\}$. Then Theorem 4.3.2 can be rewritten as:

Theorem 4.3.3. *Let G be a connected graph on $n = n_1 + n_2 + \cdots + n_k$ vertices, where $n_i \geq 2$ ($1 \leq i \leq k$), and $\delta(G) \geq \frac{n-k+\lambda}{2}$. Then G contains a spanning subgraph consisting of paths of orders $n_1, n_2, \dots, n_k$.*

In order to prove Theorem 4.3.2, we need the following two lemmas.

Lemma 4.3.1 (Enomoto, Kaneko and Tuza [187]). *Let G be a connected graph on n vertices such that $d(x) + d(y) + d(z) \geq n$ for every independent set $\{x, y, z\}$ of size three, or having no independent sets of size three. Then either G contains a Hamiltonian path or every longest cycle of G is dominating.*

Knowing Lemma 4.3.1, the main observation in proving Theorem 4.3.2 is contained in the following lemma.

Lemma 4.3.2 (Johansson [266]). *Let G be a graph with $n = n_1 + n_2 + \cdots + n_k$ vertices, where all n_i are odd positive integers. Suppose that G contains a path P such that every vertex $v \in V(G) - V(P)$ has no two consecutive neighbors on P , and satisfies $d(v \rightarrow P) \geq \lfloor \frac{1}{2}n_1 \rfloor + \lfloor \frac{1}{2}n_2 \rfloor + \cdots + \lfloor \frac{1}{2}n_k \rfloor$, where $d(v \rightarrow P)$ denotes the number of vertices of P that are adjacent to v in G . Then G contains vertex disjoint paths of orders $n_1, n_2, \dots, n_k$.*

Proof. Use induction on k . Clearly, we may assume that $|P| < n$, and that no vertex x is adjacent to an end-vertex of P (since in this case we can extend P with the vertex x). If $k = 1$, then $d(v \rightarrow P) \geq \lfloor \frac{n}{2} \rfloor > \lfloor \frac{1}{2}|P| \rfloor \geq d(v \rightarrow P)$ for every $v \in V(G) - V(P)$, a contradiction. (The last inequality follows from the fact that v is not adjacent to an end vertex of P and has no consecutive neighbors on P). Now if $k \geq 2$, let $P' \subseteq P$ be a path of order n_1 containing an end-vertex of P (P' exists since $\lfloor \frac{1}{2}|P| \rfloor > \lfloor \frac{1}{2}n_1 \rfloor$). Since n_1 is of odd order, we have $d(v \rightarrow P') \leq \lfloor \frac{1}{2}n_1 \rfloor$ and thus the graph $G' = G - P'$ and $P'' = P - P'$ satisfies the hypothesis of the lemma and G' contains paths of orders $n_2, n_3, \dots, n_k$. Then G contains vertex disjoint paths of orders $n_1, n_2, \dots, n_k$. $\square$

Proof of Theorem 4.3.2. If all the n_i 's are even, then $\delta(G) \geq \frac{n}{2}$. Thus G contains a Hamiltonian cycle by Dirac's Theorem (i.e., Theorem 4.4.1), we are done. Suppose that at least one of the n_i 's is odd. Order the n_i 's such that $n_1, n_2, \dots, n_{p-1}$ are even and $n_p, n_{p+1}, \dots, n_k$ are odd. If we put $n'_1 = n_1 + n_2 + \cdots + n_p$, and $n'_{i+1} = n_{i+p}$ for $1 \leq i \leq k - p$, then

$$\lfloor \frac{1}{2}n_1 \rfloor + \cdots + \lfloor \frac{1}{2}n_k \rfloor = \lfloor \frac{1}{2}n'_1 \rfloor + \cdots + \lfloor \frac{1}{2}n'_{k-p+1} \rfloor$$

and all the n'_i 's are odd. So we only have to consider paths of odd order. Since $n_i \geq 2$, we have $\lfloor \frac{1}{2}n_i \rfloor \geq \frac{1}{3}n_i$ and so $\delta(G) \geq \frac{1}{3}n$. Then, by Lemma 4.3.1, we may assume that G contains a dominating path P . By choosing P to be a longest dominating path, no vertex in $G - P$ has two consecutive neighbors on P . Then G and P satisfies the hypothesis of Lemma 4.3.2. This proves the theorem. $\square$

Note that the degree condition in Theorem 4.3.2 can not be weakened. Let $n = n_1 + n_2 + \cdots + n_k$ and $t = \lfloor \frac{1}{2}n_1 \rfloor + \lfloor \frac{1}{2}n_2 \rfloor + \cdots + \lfloor \frac{1}{2}n_k \rfloor - 1$. Consider the complete bipartite graph $G = K_{t, n-t}$. Then G is connected and $\delta(G) = t$.

Since each path would contribute at least $\lfloor \frac{1}{2}n_i \rfloor$ vertices to the smaller vertex class of G , G can not have a spanning subgraph consisting of vertex disjoint paths of orders $n_1, n_2, \dots, n_k$.

Recently, Chen, Tian and Wei (2001) improved this theorem, and obtain the following.

Theorem 4.3.4 (Chen, Tian and Wei [147]). *Let G be a connected graph on $n = n_1 + n_2 + \dots + n_k$ vertices, where $n_i \geq 2$ for all i , $1 \leq i \leq k$. If $n \geq 3(k - \lambda) + 4$ and $\sigma_3(G) \geq \frac{3}{2}(n - k + \lambda) - 2$, then G contains vertex disjoint paths of orders $n_1, n_2, \dots, n_k$.*

Consider the complete bipartite graph $K_{a,b}$, where $a = \frac{1}{2}(n - k + \lambda) - 1$ and $b = n - a$. Then $\sigma_3(G) = 3(n - k + \lambda)/2 - 3$. It is easy to see that $K_{a,b}$ has no vertex disjoint paths of orders $n_1, n_2, \dots, n_k$. Thus the degree condition in Theorem 4.3.4 can not be weakened.

The problem of $\{K_r\}$ -factors are well-studied. Corrádi and Hajnal (1963) gave a sufficient condition for $\{K_3\}$ -factors in terms of minimum degree. This was extended to $\{K_r\}$ -factor by Hajnal and Szemerédi (1970).

Theorem 4.3.5 (Hajnal and Szemerédi [237]). *Let G be a graph of order mr ($r \geq 2$). If $\delta(G) \geq m(r - 1)$, then G has a $\{K_r\}$ -factor.*

For $\{K_2, K_3\}$ -factors, there is a sufficient and necessary condition due to Hell and Kirkpatrick (1984). Let $\hat{c}(G)$ denote the number of critical $\{K_2, K_3\}$ -components in G (i.e, $\hat{c}(G)$ is the number of components C of G such that C has no $\{K_2, K_3\}$ -factors but $C - v$ has a 1-factor for *some* $v \in V(C)$). In Fig. 4.5, $\hat{c}(G - S) = 1$.

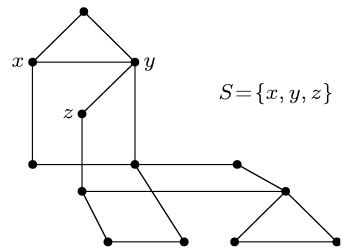


Fig. 4.5 $\hat{c}(G - S) = 1$.

Theorem 4.3.6 (Hell and Kirkpatrick [243]). *A graph G has a $\{K_2, K_3\}$ -factor if and only if*

$$\hat{c}(G - S) \leq |S|$$

for all $S \subset V(G)$.

Of course, this factor can also be stated as a $\{P_2, C_3\}$ -factor. So it can be placed in Section 4.2 too.

Let K_4^- denote the complete graph K_4 with one edge removed. According to Theorem 4.0.1, there unlikely exists a sufficient and necessary condition for K_4^- -factors. Kawarabayashi (2002) obtained a simple sufficient condition in term of minimum degree.

Theorem 4.3.7 (Kawarabayashi [306]). *Let G be a graph of order $4m$. If $\delta(G) \geq 5m/2$, then G has a K_4^- -factor.*

Lonc and Ryjáček [381] introduced a new type of component factor which has a restrictive occurrence on one component. Let H and F be graphs. An H - $\{F\}$ -factor is a spanning subgraph such that exactly one component is isomorphic to H and all the other components are isomorphic to F . They obtained a complete characterization for classes of claw-free graphs of odd order that fail to have $K_{3-}\{P_2\}$ -factors, or $P_{3-}\{P_2\}$ -factors, or $C_{2t+1-}\{P_2\}$ -factors.

4.4 Connected $[a, b]$ -factors

One of new trend in factor theory is to study factors with an additional property. In this section and the next two sections, we shall put our emphasis on factors with connectivity condition, i.e., connected factors. Some of the earliest results of graph theory on Hamiltonian cycles, trees and walks fall naturally into this category. In the last decade the notion of connected factors has gained wider acceptance.

Let g, f be the nonnegative integer functions defined on vertex set $V(G)$ of a graph G and let F be a (g, f) -factor of G . If F is connected, then we call F a *connected (g, f) -factor*. If $g(v) = a, f(v) = b$ for every vertex $v \in V(G)$, then F is called a *connected $[a, b]$ -factor* of G . In particular, when $a = b = k$, then F is a *connected k -factor*.

Clearly, G has a (g, f) -factor, but G may not necessarily have a connected (g, f) -factor. For example, Petersen graph contains a 2-factor, but it does not contain a connected 2-factor, i.e. a Hamiltonian cycle. The problem of determining whether a graph has a (g, f) -factor has a polynomial algorithm, but determining if a graph has a connected (g, f) -factor is NP-complete (see [225]).

Let G be a graph. A cycle C of G is *Hamiltonian* if it passes once and only once through every vertex of G ; likewise, a path P is *Hamiltonian* if it passes every vertex exactly once. Obviously, a Hamiltonian cycle is a connected 2-factor and a Hamiltonian path is a connected $[1, 2]$ -factor.

Since determining whether a graph is Hamiltonian is an NP-complete problem, we are interested in finding a nontrivial sufficient condition for a graph

to have a Hamiltonian cycle. The following well-known theorem was given by Dirac (1952), which is a sufficient condition for the existence of Hamiltonian cycles. This theorem has lead to many conjectures and theorems concerning cycles and paths.

Theorem 4.4.1 (Dirac [161]). *Let G be a graph of order n . If $\delta(G) \geq \frac{n}{2}$, then G contains a Hamiltonian cycle.*

Ore (1960) extended the method used in the above theorem and obtained the following stronger result.

Theorem 4.4.2 (Ore [470]). *Let G be a graph of order n . If $\sigma_2(G) \geq n$, then G contains a Hamiltonian cycle.*

The condition “ $\delta(G) \geq n/2$ ” is called “Dirac’s condition” and “ $\sigma_2(G) \geq n$ ” is referred as “Ore’s condition”. Those two conditions play important roles in investigating connected factors.

Let $P = v_1v_2 \dots v_k$ be a longest path of a 2-connected graph G . We denote by $\text{End}(P)$ the set of end-vertices of P , i.e., $\text{End}(P) = \{v_1, v_k\}$. By the maximality of P , we easily have the following lemma.

Lemma 4.4.1.

- (a) $N(v_1), N(v_k) \subseteq V(P)$;
- (b) if $v_1v_k \in E(G)$, then G is Hamiltonian;
- (c) if $v_1v_j \in E(G)$, then $Q = (P \cup v_1v_j) - v_{j-1}v_j$ is a longest path of G such that $\text{End}(Q) = \{v_{j-1}, v_k\}$.

Lemma 4.4.2 (Bondy [86]). *Let $P = v_1v_2 \dots v_k$ be a longest path in a 2-connected graph G of order n . Then $c(G) \geq \min\{n, d(v_1) + d(v_k)\}$, where $c(G)$ is the circumference of G .*

Now we prove the following theorem, known as Fan’s Theorem, using Lemmas 4.4.1 and 4.4.2. The proof here is from [548].

Theorem 4.4.3 (Fan [193]). *Let p be a positive integer and G be a 2-connected graph of order n . If $\max\{d(u), d(v)\} \geq \frac{p}{2}$ for any two vertices u and v with $d(u, v) = 2$, then $c(G) \geq p$.*

Proof. Let $S = \{v \mid d(v) \geq p/2\}$. By Lemma 4.4.2, it is enough to prove that there exists a longest path P in G such that $\text{End}(P) \subseteq S$.

Choose the longest path $P = v_1v_2 \dots v_k$ of G such that

$$|\text{End}(P) \cap S| \text{ is maximized.} \quad (*)$$

Assume, without loss of generality, that $v_1 \notin S$. Denote by $\mathcal{P}$ the set of all longest path of G having v_k as an end-vertex. Let $Q = q_1q_2 \dots q_k$ be any element of $\mathcal{P}$. Clearly $q_k = v_k$. Set $\lambda(Q, q_1) = \max\{i \mid q_i \in N(q_1)\}$. By the choice (*), we have

$$q_1 \notin S. \quad (4.3)$$

By Lemma 4.4.1 and the 2-connectivity of G , we have $\lambda(Q, q_1) \geq 3$. We may assume that $\lambda(Q, q_1) \leq k - 1$; otherwise $c(G) = n$ by Lemma 4.4.1 and we are done. Thus we have

$$3 \leq \lambda(Q, q_1) \leq k - 1. \quad (4.4)$$

Now we prove

$$\{q_2, q_3, \dots, q_m\} \subseteq N(q_1), \quad (4.5)$$

where $m = \lambda(Q, q_1)$. Otherwise, there must be a vertex q_t , $4 \leq t \leq m$, such that $q_1 q_t \in E(G)$, but $q_1 q_{t-1} \notin E(G)$. By Lemma 4.4.1 and the choice (*), we have $q_{t-1} \notin S$. Since $d(q_1, q_{t-1}) = 2$ and $q_1 \notin S$, it follows $\max\{d_G(q_1), d_G(q_{t-1})\} < p/2$, a contradiction.

We now choose $P' = v_1 v_2 \dots v_k \in \mathcal{P}$ such that

$$\lambda(P', v_1) \text{ is maximized.} \quad (**)$$

Claim. $N(v_j) \subseteq \{v_1, v_2, \dots, v_m\}$, for each $j = 1, 2, \dots, m - 1$, where $m = \lambda(P', v_1)$.

By (4.5) and Lemma 4.4.1, we have $N(v_j) \subseteq V(P')$ for each $j = 1, 2, \dots, m - 1$. Assume that the claim does not hold. Thus there exist two vertices v_r and v_s with $1 \leq r \leq m - 1$ and $m + 1 \leq s \leq k$, such that $v_r v_s \in E(G)$. By (4.5), we have $v_1 v_{r+1} \in E(G)$. Thus $Q = (P' \cup v_1 v_{r+1}) - v_r v_{r+1} \in \mathcal{P}$ with $\lambda(Q, v_r) \geq s > m = \lambda(P', v_1)$, which contradicts the choice (**). Claim is proved.

Now, by Claim above, we see that v_m is a cutvertex of G , which contradicts the 2-connectivity of G . $\square$

In 1972, Chvátal and Erdős [150] pioneered the research of connectivity and independent number for the existence of Hamiltonian cycles, and obtained the following nice result.

Theorem 4.4.4 (Chvátal and Erdős [150]). *Let G be a graph with at least three vertices. If G is κ -connected and $\alpha(G) \leq \kappa$, then G has a Hamiltonian cycle.*

Proof. Let G be a graph satisfying the hypothesis of the theorem. Clearly, G contains a cycle and let C be the longest one. If C is not a Hamiltonian cycle, then there exists a vertex $x \notin V(C)$. Since G is κ -connected, by Menger's Theorem, there are κ internally-disjoint paths starting at x and terminating in C which share with C only their terminal vertices $x_1, x_2, \dots, x_\kappa$. For each $i = 1, 2, \dots, \kappa$, let y_i be the successor of x_i in a fixed cyclic ordering of C , then no y_i is adjacent to x . Otherwise, we would replace the edge $x_i y_i$ in C by the path going from x_i to y_i outside C (via x) and obtain a longer

cycle. However, G contains no independent sets of $\kappa + 1$ vertices and so there is an edge $y_i y_j$. Delete the edge $x_i y_i, x_j y_j$ from C and add the edge $y_i y_j$ together with the path going from x_i to x_j outside C . In this way, we obtain a cycle longer than C , a contradiction again. Therefore, the longest cycle C is Hamiltonian. $\square$

Corollary 4.4.1. *Let G be a graph with at least three vertices. If G is κ -connected and $\alpha(G) \leq \kappa + 1$, then G has a Hamiltonian path.*

Due to the limited space, we only mention very selective results above. There are extensive literatures on the subject of Hamiltonian cycles. The reader can find many books and survey papers covering this topic, e.g., [228], [229].

Hamiltonian cycle problem is one of most well-studied subject in graph theory. Since Hamiltonian cycles can be regarded as connected 2-factors, many ideas and approaches can be extended from the theory of Hamiltonian cycles to the study of connected factors. Based on this, Kano (1994) posed the following general problems about connected factors.

Open Problem 4.4.1 (Kano [281]). *Find sufficient conditions for graphs to have connected $[a, b]$ -factors from known results on Hamiltonian cycles or paths. In particular, sufficient conditions for a graph G to have a connected $[k, k + 1]$ -factor.*

For connected $[2, k]$ -factors, Kano [282] and Kouider and Maheo [324] obtained a sufficient condition in terms of degree sum.

Theorem 4.4.5 (Kano [282]; Kouider and Maheo [324]). *Let k be an integer with $k \geq 2$, and let G be a connected graph of order n with $\delta(G) \geq 2$. If $n \geq k + 3$ and $\sigma_2(G) \geq 4n/(k + 2)$, then G has a connected $[2, k]$ -factor.*

Further from Theorem 4.4.5, Kano asked whether the condition $\sigma_2(G) \geq 4n/(k + 2)$ is enough for the existence of a 2-edge-connected $[2, k]$ -factor.

When k is odd, Xu [593] constructed a class of graphs satisfying the condition $\sigma_2(G) \geq 4n/(k + 2)$ but have no 2-edge-connected $[2, k]$ -factors. Xu's construction goes like this: choose positive integers m and n such that $k = 2m - 1$, $m \geq 2$ and $n \geq m(2m + 1)$. Then there are positive integers $n_1, n_2, \dots, n_m$ such that $n_1 \leq n_2 \leq \dots \leq n_m$, $n = n_1 + n_2 + \dots + n_m + 1$ and $n_m - n_1 \leq 1$. Denoted by K_{n_i} the complete graph with n_i vertices and $G' = K_{n_1} \cup K_{n_2} \cup \dots \cup K_{n_m}$. Now let $G = G' \vee w$, i.e., G is obtained by joining a new vertex w to every vertex of G' .

It is clear that $\delta(G) = n_1 \geq 4$ and $|V(G)| = n \geq m(2m + 1) > 2m + 2 = k + 3$ since $m \geq 2$. Moreover, it is not hard to show that G is a n_1 -edge-connected graph and thus G is at least 4-edge-connected.

It remains to show that the condition $\sigma_2(G) \geq 4n/(k + 2)$. For any two vertices x and y of G , we have $d(x) + d(y) \geq 2\delta(G) = 2n_1$. Clearly, there

is l with $1 \leq l \leq m$ such that $n = mn_1 + l$. Then $n_1 = (n - l)/m$. So $d(x) + d(y) \geq 2(n - l)/m$. Set $w = 2(n - l)/m - 4n/(k + 2)$. Then

$$\begin{aligned}
 w &= 2(n - l)/m - 4n/(2m + 1) \\
 &= [2(n - l)(2m + 1) - 4mn]/m(2m + 1) \\
 &= [2n - 2l(2m + 1)]/m(2m + 1) \\
 &\geq [2m(2m + 1) - 2l(2m + 1)]/m(2m + 1) \\
 &= 2(m - l)(2m + 1)/m(2m + 1) \\
 &= 2(m - l)/m,
 \end{aligned}$$

which implies that $d(x) + d(y) \geq 4n/(k + 2)$. Next to see that G has no 2-edge-connected $[2, k]$ -factors. Otherwise, if G has a 2-edge-connected $[2, k]$ -factor F , then $|N_F(x) \cap K_{n_i}| \geq 2$ for $i = 1, 2, \dots, m$, and thus $d_F(x) \geq 2m > 2m - 1 = k$, a contradiction. (Note that Li and Cai [346] also come up an example to show the insufficiency of the condition $\sigma_2(G) \geq 4n/(k + 2)$ for the existence of 2-edge-connected $[2, k]$ -factors.)

Despite the examples above for odd k , we do have a positive answer if relaxing the range of factor a little bit. For even k , Kouider and Maheo [324] showed that the condition $\sigma_2(G) \geq 4n/(k + 2)$ is enough for the existence of 2-edge-connected $[2, k]$ -factors.

Theorem 4.4.6 (Kouider and Maheo [324]). *If G is a 2-edge-connected graph of order n , $k \geq 2$ and $n \geq k + 3$. If $\sigma_2(G) \geq 4n/(k + 2)$, then*

- (a) *k is even, G has a 2-edge-connected $[2, k]$ -factor;*
- (b) *k is odd, G has a 2-edge-connected $[2, k + 1]$ -factor.*

Note that for $k = 2$, the above theorem is implied by Theorem 4.4.2.

Combining with $\sigma_2(G)$ condition and strengthening the condition to 2-connectivity from 2-edge-connectivity in hoping to be enough for the existence of 2-edge-connected $[2, k]$ -factors, Xu (1999) posed the following conjecture.

Conjecture 4.4.1 (Xu [593]). *Let k be an odd integer and G a 2-connected graph of order n with $\delta(G) \geq 2$ and $n \geq k + 3$. If $\sigma_2(G) \geq 4n/(k + 2)$, then G has a 2-edge-connected $[2, k]$ -factor.*

For a more general discussion, i.e., connected $[a, b]$ -factors, Kouider and Mahéo (2002) obtained the following nice result.

Theorem 4.4.7 (Kouider and Mahéo [325]). *Let G be a connected graph of order n and let a and b be two integers such that $2 \leq a \leq b$, a divides b . If the conditions (i) $n \geq (a + b)(a + b - 1)/b$ and (ii) $\delta(G) \geq an/(a + b)$ are satisfied, then G has a connected $[a, b]$ -factor.*

Note that in Theorem 4.4.7, there is a condition that a divides b . Kouider and Mahéo pointed out that this condition can be easily eliminated by modifying the other condition slightly.

Theorem 4.4.8 (Kouider and Mahéo [325]). *Let a and b be two integers such that $4 \leq 2a < b$ and G a connected graph of order n . If $n \geq \frac{(a+b)(a+b-1)}{b}$ and $\delta(G) \geq \frac{n}{1+\lfloor \frac{b}{a} \rfloor}$, then G has a connected $[a, b]$ -factor.*

Proof. Let $b_1 = a \lfloor \frac{b}{a} \rfloor$. Then $2a \leq b_1 < b$ and a divides b_1 . On the other hand, $\delta(G) \geq \frac{an}{a+b_1}$ and $n \geq \frac{(a+b)(a+b-1)}{b} \geq \frac{(a+b_1)(a+b_1-1)}{b_1}$.

From Theorem 4.4.7, G has a connected $[a, b_1]$ -factor, which is also a connected $[a, b]$ -factor. $\square$

Example 4.4.1. *Let a and b be two integers such that a divides b and $q = \frac{b}{a}p + 1$, where $p \geq 1$. Consider the graph $K_{p,q}$ and note that its minimum degree is equal to p and $p < \frac{aq}{b} < \frac{an}{a+b}$. On the other hand, if F is a connected $[a, b]$ -factor in this graph, it is clear that $qa \leq |E(F)| \leq pb$, which contradicts the choice of p and q . So, $K_{p,q}$ has no $[a, b]$ -factors.*

This example shows that the bound on minimum degree in Theorem 4.4.7 is best possible. Xu (1999) extended Theorem 4.4.7 partially from the minimum degree condition to $\sigma_2(G)$ type condition for the case of $a \leq b \leq 2a$.

Theorem 4.4.9 (Xu [593]). *Let a and b be two integers with $6 \leq a \leq b \leq 2a$ and let G be a connected graph with $|V(G)| \geq \max\{25, \frac{(a+b)(a+b-1)}{b}\}$. If $\sigma_2(G) \geq \frac{2an}{a+b}$, then G contains a connected $[a, b+1]$ -factor.*

For connected $[k, k+1]$ -factor, Cai (1997) obtained the following theorem and his proof is quite involved.

Theorem 4.4.10 (Cai [105]). *Let k be an odd integer with $k \geq 3$ and G be a connected graph of odd order n with $n \geq 4k - 3$, and $\delta(G) \geq k$. If $\max\{d(u), d(v)\} \geq \frac{n}{2}$ for every pair of nonadjacent vertices u, v of G , then G has a connected $[k, k+1]$ -factor.*

Next, we consider the sufficient conditions for connected factors in terms of graphic parameters, such as independent number $\alpha(G)$, toughness $t(G)$, connectivity, etc. Brandt [90] obtained the following theorem in terms of connectivity and independent number.

Theorem 4.4.11 (Brandt [90]). *Let b and k be integers, where b is even and $k \geq 2$. If a graph G is k -connected and $\alpha(G) \leq \frac{kb}{2}$, then G has a 2-connected $[2, b]$ -factor.*

Recently, Kouider and Lonc (2004) also considered the connected $[a, b]$ -factor under the condition of connectivity and independent number.

Theorem 4.4.12 (Kouider and Lonc [323]). *Let a and b be two positive integers with $b \geq a + 1$ and one of the following conditions holds:*

- (a) $a \geq 4$ and $(a, b) \neq (4, 7)$;
- (b) $a = 3$ and $b \equiv 0 \pmod{4}$.

If G is k -connected with $k \geq 2$, $\delta(G) \geq 2k + a$ and $\alpha(G) \leq \frac{4bk}{(a+1)^2}$, then G has a connected $[a, b]$ -factor.

For the other results about connected factors, the readers are referred to the original articles, e.g., [339], [553] or the survey paper [328].

To conclude this section, we pose the following conjectures.

Conjecture 4.4.2. *Let k be a positive integer and let G be a graph of order n with $n \geq k + 1$. If $t(G) \geq k$, then G has a connected $[k, k + 1]$ -factor.*

The more general setting of the above conjecture is the following.

Open Problem 4.4.2. *Find the smallest integer c such that a connected graph G of order $n \geq k + 1$ and $t(G) \geq k$ has a connected $[k, k + c]$ -factor.*

Conjecture 4.4.3. *Let G be a connected graph of order n such that $\delta(G) \geq k$, $k \geq 2$, kn even and $n \geq 4k - 3$. If $\sigma_2(G) \geq n$, then G has a connected k -factor.*

The case of $k = 2$ is true by well-known Ore's theorem. Enomoto *et al.* [186] have proved that such a graph has a k -factor. It follows from Theorem 4.4.2 that such a graph has a connected $[k, k + 1]$ -factor.

4.5 Connected (g, f) -factors

In the previous section, we discussed the connected $[a, b]$ -factors and presented some of progresses in that direction. In this section, we consider connected (g, f) -factors. The main idea is to adjust an existing (g, f) -factor through a tree or other spanning subgraphs to obtain a connected (g, f) -factor.

Let f be a positive integer-valued function defined on vertex set $V(G)$. If an f -factor of G is a tree, we call it f -tree. In particular, an f -tree with maximum degree k is called a k -tree.

Theorem 4.5.1 (Tokuda, Xu and Wang [553]). *Let G be a graph, and g, f, f' be positive integer-valued functions defined on $V(G)$. If G has a (g, f) -factor and an f' -tree, then G contains a connected $(g, f + f' - 1)$ -factor.*

Proof. Let F be a (g, f) -factor of G , and let $F_1, F_2, \dots, F_m$ be the components of F such that $d_F(x) = f(x)$ for all $x \in V(F_i)$. For each $1 \leq i \leq m$, choose one edge $u_i v_i$ of F_i so that v_i is not a cutvertex of F_i .

Denote the set of triples (H, D, T) by $\mathcal{F}$, where D is a subset of $\{u_1 v_1, \dots, u_m v_m\}$, T is a tree of G (not necessary to be an f' -tree), and H is a connected

$(g-1, f+f')$ -factor of G such that $E(T) \cap D = \emptyset$ and $E(H) = E(F-D) \cup E(T)$ with the following conditions:

- $d_H(x) \leq f(x) + f'(x) - 1$ for all $x \in V(D)$;
- $N_F(v) \cap N_T(v) = \emptyset$ for all $v \in V(D) \cap \{v_1, v_2, \dots, v_m\}$;
- For each $x \in V(H) - V(D)$, if $d_H(x) = f(x) + f'(x)$, then $N_F(x) \cap N_T(x) = \emptyset$ and $d_F(x) = f(x)$.

Note that if we choose T, D and H to be an f' -tree of G , an empty set and a connected factor of G induced by $E(F) \cup E(T)$, respectively, then (H, D, T) satisfies the above conditions. Thus $\mathcal{F} \neq \emptyset$. Fig. 4.6 gives an illustration of T, D and H .

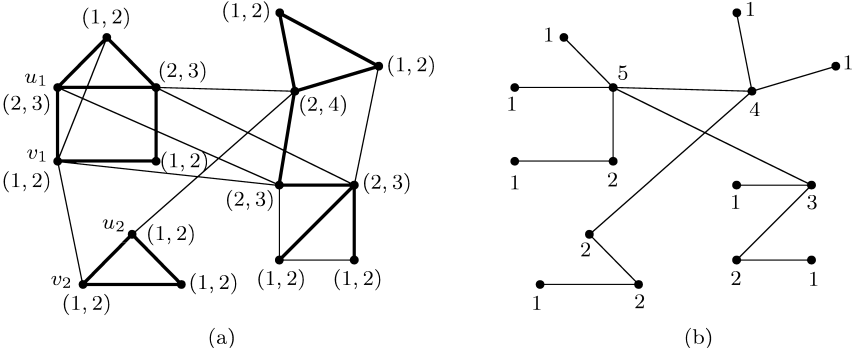


Fig. 4.6 (a) $D = \emptyset$, (g, f) -factor is in bold lines, where g, f are labeled on vertices; (b) f' -tree, where f' is labeled on vertices.

Claim 1. $d_H(x) \geq g(x)$ for all $x \in V(G) - (V(D) \cap \{u_1, u_2, \dots, u_m\})$.

Since $E(H) = E(F - D) \cup E(T)$ for all $x \in V(G) - V(D)$, $d_H(x) \geq d_{F-D}(x) = d_F(x) \geq g(x)$. For all $v \in V(D) \cap \{v_1, v_2, \dots, v_m\}$, $N_F(v) \cap N_T(v) = \emptyset$ by the definition of $\mathcal{F}$. Therefore $d_H(v) = d_{F-D}(v) + d_T(v) \geq g(v) - 1 + 1 = g(v)$.

Here we choose $(H, D, T) \in \mathcal{F}$ such that

- (a) $|E(H)|$ is as small as possible;
- (b) $|D|$ is as small as possible, subject to be at least 1.

Then we prove the following two more claims.

Claim 2. $d_H(x) \leq f(x) + f'(x) - 1$ for all $x \in V(G)$.

Let F_C be a component of F . Assume that $d_H(x) = f(x) + f'(x)$ for all $x \in V(F_C)$. By the definition of $\mathcal{F}$, $E(F_C) \cap D = \emptyset$, $N_F(x) \cap N_T(x) = \emptyset$ and $d_F(x) = f(x)$ for all $x \in V(F_C)$. Consequently, there is some integer k such that $u_k v_k \in E(F_C)$, $u_k v_k \notin E(T)$ and $u_k v_k \notin D$. Then we can choose $D' = D \cup \{u_k v_k\}$ and $H' = H - u_k v_k$ such that $(H', D', T) \in \mathcal{F}$ and

$|E(H')| < |E(H)|$, which contradicts to the choice of (H, D, T) . Hence there is a vertex x of $V(F_C)$ such that $d_H(x) \leq f(x) + f'(x) - 1$.

Next, suppose that $V(F_C)$ contains a vertex y with $d_H(y) = f(y) + f'(y)$. If $E(F_C) \cap D = \{u_k v_k\}$ for some integer k , then $F_C - v_k$ has an edge $x_0 y_0$ on $u_k y$ -path with $d_H(x_0) \leq f(x_0) + f'(x_0) - 1$ and $d_H(y_0) = f(y_0) + f'(y_0)$ since $F_C - v_k$ is connected (note that v_k is not a cutvertex), $d_H(u_k) \leq f(u_k) + f'(u_k) - 1$ and $d_H(y) = f(y) + f'(y)$. Similarly, if $E(F_C) \cap D = \emptyset$, then F_C contains an edge $x_0 y_0$ on xy -path with $d_H(x_0) \leq f(x_0) + f'(x_0) - 1$ and $d_H(y_0) = f(y_0) + f'(y_0)$. By the definition of $\mathcal{F}$ and $d_H(y_0) = f(y_0) + f'(y_0)$, $N_F(y_0) \cap N_T(y_0) = \emptyset$. Since $x_0 \in N_F(y_0)$, $x_0 y_0 \notin E(T)$. Because T is a spanning tree, $T \cup x_0 y_0$ contains a cycle C . Let $\{w\} = N_C(y_0) - \{x_0\}$. Then $y_0 w \in E(C - x_0 y_0) \subset E(T)$ and $y_0 w \notin E(F_C)$. Choose $T' = T \cup x_0 y_0 - y_0 w$ and $H' = H - y_0 w$. Then it is easy to see that $x_0 \notin V(D) \cap \{v_1, v_2, \dots, v_m\}$, $g(x_0) - 1 \leq d_{H'}(x_0) = d_H(x_0) \leq f(x_0) + f'(x_0) - 1$, $d_{H'}(y_0) = d_H(y_0) - 1 = f(y_0) + f'(y_0) - 1$ and $d_{H'}(w) = d_H(w) - 1 \leq f(w) + f'(w) - 1$. Next, we consider a vertex w in H' . If $w \notin V(D)$, then $g(w) \leq d_F(w) \leq d_{H'}(w)$. If $w \in V(D) \cap \{v_1, v_2, \dots, v_m\}$, then $d_{H'}(w) = d_{F-D}(w) + d_{T'}(w) \geq g(w)$. And if $w \in V(D) \cap \{u_1, u_2, \dots, u_m\}$, then $d_{H'}(w) \geq d_{F-D}(w) \geq g(w) - 1$. Hence $(H', D, T') \in \mathcal{F}$ and $|E(H')| < |E(H)|$, a contradiction.

Claim 3. $d_H(u) \geq g(u)$ for all $u \in V(D) \cap \{u_1, u_2, \dots, u_m\}$.

Suppose D contains some edge $uv = u_i v_i$ with $d_H(u) = g(u) - 1$. Then $uv \in E(F)$ and $uv \notin E(T)$ since $D \subset E(F)$ and $E(T) \cap D = \emptyset$. Hence tree $T \cup uv$ has a cycle C . Let $\{w\} = N_C(v) - \{u\}$. Then $vw \in E(C - uv) \subset E(T)$ and $vw \notin E(F)$ because $N_F(v) \cap N_T(v) = \emptyset$. Choose $D' = D - \{uv\}$, $T' = T \cup uv - vw$ and $H' = H \cup uv - vw$. Then it is easy to see that vertices $u, v \notin V(D')$, $g(v) \leq d_{H'}(v) = d_H(v) \leq f(v) + f'(v) - 1$ and $d_{H'}(u) = d_H(u) + 1 = g(u)$. Next, consider a vertex w in H' . If $w \notin V(D')$, then $d_{H'}(w) \geq d_F(w) \geq g(w)$. If $w \in V(D') \cap \{v_1, v_2, \dots, v_m\}$, then $d_{H'}(w) = d_{F-D'}(w) + d_{T'}(w) \geq g(w)$. And if $w \in V(D') \cap \{u_1, u_2, \dots, u_m\}$, then $d_{H'}(w) \geq d_{F-D'}(w) \geq g(w) - 1$. Hence $(H', D', T') \in \mathcal{F}$ with $|E(H')| = |E(H)|$ and $|D'| < |D|$. This contradicts to the choice of (H, D, T) .

By Claims 1, 2 and 3, H is a connected $(g, f + f' - 1)$ -factor of G . $\square$

Li *et al.* (2005) considered the graphs containing both a (g, f) -factor and a Hamiltonian path shown in the theorem.

Theorem 4.5.2 (Li *et al.* [339]). *Let G be a connected graph, let g, f be two positive integer-valued functions from $V(G)$ to $\mathbb{Z}^+ - \{1\}$ such that $g(v) \leq f(v) \leq d(v)$ for all $v \in V(G)$. If G has both a (g, f) -factor and a Hamiltonian path, then G contains a connected $(g, f + 1)$ -factor.*

By posing an additional condition, such as $K_{1,n}$ -freedom, as a sufficient condition, many connected factor results are obtained.

Theorem 4.5.3 (Li and Liu [337]). *Let G be a connected claw-free graph. Then G has a connected $[1, 3]$ -factor. Moreover, if G is also 2-connected, then it has $[2, 3]$ -factor.*

Xu, Liu and Tokuda (1998) considered the connected (g, f) -factor in $K_{1,n}$ -free graphs and obtained the following theorem.

Theorem 4.5.4 (Xu, Liu and Tokuda [592]). *Let G be a connected $K_{1,n}$ -free graph ($n \geq 3$), f and g be positive integer-valued functions defined on $V(G)$ with $g(v) \leq f(v)$ for all $v \in V(G)$. If G has a (g, f) -factor, then G contains a connected $(g, f + n - 1)$ -factor.*

To show Theorem 4.5.4, we introduce a few new terms.

Let F be a (g, f) -factor of $K_{1,n}$ -free graph G , and $W_1, W_2, \dots, W_m$ be all components of F . Let $\psi(G) = G / \bigcup_{i=1}^m W_i$ be the graph obtained from G by contracting each component W_i into a vertex v_{W_i} for $i = 1, 2, \dots, m$ and removing loops resulted from the contracting procedure. A subgraph T of G is called *F-factor tree* if T is a connected graph which is formed by k components of F and $k - 1$ edges of G . It is easy to see that $\psi(T)$ is indeed a tree.

Let $\phi(F)$ be the set of all F -factor trees such that for each $T \in \phi(F)$, $W_1 \subseteq T$, $d_T(x_T) \leq f(x_T) + n$ for a vertex $x_T \in V(T)$, and $d_T(y) \leq f(y) + n - 1$ for all $y \in V(T) - \{x_T\}$. It is easy to see that $\phi(F) \neq \emptyset$, e.g., $W_1 \in \phi(F)$. For each $T \in \phi(F)$, denote by W_T the component of F with $x_T \in V(W_T)$, and define

$$\text{dist}(T) = \begin{cases} d_{\psi(T)}(v_{W_1}, v_{W_T}) & \text{if } d_T(x_T) = f(x_T) + n; \\ \infty & \text{if } d_T(x_T) < f(x_T) + n. \end{cases}$$

Let $\phi'(F)$ be the subset of $\phi(F)$ such that $T' \in \phi'(F)$ implies that $|V(T')| \geq |V(T)|$ for all $T \in \phi(F)$. We choose an F -factor tree $T_0 \in \phi'(F)$ such that $\text{dist}(T_0) \geq \text{dist}(T)$ for all $T \in \phi'(F)$ and use $x = x_{T_0}$ and $W = W_{T_0}$ for convenience.

Now we are ready to provide a proof of Theorem 4.5.4.

Proof of Theorem 4.5.4. Assume that G has no connected $(g, f + n - 1)$ -factors.

Claim 1. $\text{dist}(T_0) < \infty$. That is, $d_{T_0}(x) = f(x) + n$.

Assume that $d_{T_0}(x) < f(x) + n$. If $V(T_0) = V(G)$, then T_0 is a connected $(g, f + n - 1)$ -factor in G , a contradiction. Hence, we may assume that $V(T_0) \neq V(G)$. Since G is connected, let W' be a component of F such that $W' \not\subseteq T_0$ and $E_G(W', T_0) \neq \emptyset$, and e be an edge such that $e \in E_G(W', T_0)$. Let $T'_0 = T_0 \cup W' \cup e$. Then $T'_0 \in \phi(F)$ and $|V(T'_0)| > |V(T_0)|$, a contradiction to the choice of $\phi'(F)$.

By Claim 1 and the definitions of g and f , we have $|N_W(x)| \geq 1$ and $|N_{T_0-W}(x)| \geq n$. Let $x' \in N_W(x)$ and $\{u_0, u_1, \dots, u_{n-1}\} \subseteq N_{T_0-W}(x)$. Suppose that u_i is not on a v - x path in T_0 for $v \in V(W_1)$, $i = 1, 2, \dots, n-1$. Since G is $K_{1,n}$ -free and T_0 is an F -factor tree, there exists a pair of vertices u_i, u_j such that $u_i u_j \in E(G)$ and $u_i u_j \notin E(T_0)$. Let $T'_0 = T_0 \cup u_i u_j - x u_i$. Then $T'_0 \in \phi'(F)$ and $\text{dist}(T'_0) > \text{dist}(T_0)$, a contradiction to the choice of T_0 . Therefore, we may assume that u_0 is on a v - x path in T_0 .

If there exists a pair of vertices u_i, u_j with $u_i u_j \in E(G)$ for $1 \leq i, j \leq n-1$, let $T'_0 = T_0 \cup u_i u_j - x u_i$, then $T'_0 \in \phi'(F)$ and $\text{dist}(T'_0) > \text{dist}(T_0)$. Again this is a contradiction to the choice of T_0 . Hence $u_i u_j \notin E(G)$ for $i, j = 1, 2, \dots, n-1$. Since G is $K_{1,n}$ -free, $G[\{x, x'\} \cup \{u_1, u_2, \dots, u_{n-1}\}]$ is not $K_{1,n}$. Therefore there exists a vertex $u \in \{u_1, u_2, \dots, u_{n-1}\}$ with $u x' \in E(G)$.

If $d_{T_0}(x') < f(x') + n - 1$, let $T'_0 = T_0 \cup u x' - u x$, then $T'_0 \in \phi'(F)$ and $\text{dist}(T'_0) > \text{dist}(T_0)$, a contradiction. Hence $d_{T_0}(x') = f(x') + n - 1$. Then $|N_{T_0-W}(x')| \geq n - 1$. Let $\{u'_1, u'_2, \dots, u'_{n-1}\} \subseteq N_{T_0-W}(x')$. Since T_0 is an F -factor tree, $u'_i u'_j, u'_i u \notin E(T_0)$ for $i, j = 1, 2, \dots, n-1$.

If there exists a pair of vertices u_i, u_j with $u'_i u'_j \in E(G)$, let $T'_0 = T_0 \cup u'_i u'_j - x' u'_i$, then $T'_0 \in \phi'(F)$ and $\text{dist}(T'_0) > \text{dist}(T_0)$, a contradiction. Hence $u'_i u'_j \notin E(G)$ for $i, j = 1, 2, \dots, n-1$. Since G is $K_{1,n}$ -free, $G[\{x', u\} \cup \{u'_1, u'_2, \dots, u'_{n-1}\}]$ is not $K_{1,n}$. Therefore there exists a vertex $u' \in \{u'_1, u'_2, \dots, u'_{n-1}\}$ with $u' u \in E(G)$. Let $T'_0 = T_0 \cup u u' - u x$. Then $T'_0 \in \phi'(F)$ and $\text{dist}(T'_0) > \text{dist}(T_0)$, a contradiction again.

Therefore, G has a connected $(g, f + n - 1)$ -factor. $\square$

Corollary 4.5.1. *Let n, a, b be positive integers with $n \geq 3$ and $a \leq b$ and let G be a $K_{1,n}$ -free graph. If G has an $[a, b]$ -factor, then G contains a connected $[a, b + n - 1]$ -factor.*

In Theorem 4.5.4, there is no restriction on functions g and f . By restricting the range of g and f , Tokuda (1999) obtained the following stronger result.

Theorem 4.5.5 (Tokuda [551]). *Let n, a, b be positive integers with $n \geq 3$, $a \geq 1$ and $b \geq a(n - 2) + 2$. If a connected $K_{1,n}$ -free graph G has an $[a, b]$ -factor, then G has a connected $[a, b]$ -factor too.*

4.6 Generalized trees

It is well-known that every connected graph G has a tree. Every tree of G is also a connected $[1, \Delta(G)]$ -factor. Thus G has a connected $[1, k]$ -factor if and only if G has a k -tree.

Recall that a k -tree T of a graph G is a tree with $d_T(x) \leq k$ for all $x \in V(G)$. A tree T is called a k -ended tree if T has at most k -leaves. In this section, we study problems related to spanning trees, such as k -trees, k -ended trees and other trees with given properties.

Chvátal and Erdős [150] showed that if $\alpha(G) \leq \kappa(G) + 1$, then G has a Hamilton path. Since Hamiltonian paths can be considered as 2-ended trees, Las Vergnas made a conjecture about k -ended trees based on Chvátal and Erdős condition, which was confirmed by Win (1979).

Theorem 4.6.1 (Win [585]). *If $\alpha(G) \leq \kappa(G) + k - 1$, then G has a k -ended tree. Moreover, if $\alpha(G) \leq \kappa(G) + 2$, then G has a 3-ended tree which contains a longest path of G .*

There are many other sufficient conditions for existence of k -ended trees, in terms of minimum degree, degree-sum, degree-union, claw-freedom and likewise. Most of them are inspired by the sufficient conditions of Hamiltonian paths, we do not list them all here.

The *leaf distance* of a tree T is the minimum distance between any two leaves of T . The *leaf degree* of a vertex v in a tree T is the number of leaves adjacent to v in T . Kaneko (2001) gave a characterization of trees with a fixed leaf degree using the number of isolated vertices.

Theorem 4.6.2 (Kaneko [268]). *Let G be a graph and $m \geq 1$ an integer. Then G has a tree with maximum leaf degree at most m if and only if*

$$i(G - S) < (m + 1)|S| \quad \text{for all } \emptyset \neq S \subseteq V(G).$$

For leaf distance, Kaneko made the following conjecture.

Conjecture 4.6.1 (Kaneko [268]). *Let $d \geq 3$ be an integer and G a connected graph of order at least $d + 1$. If*

$$i(G - S) < \frac{2|S|}{d - 2} \quad \text{for all } \emptyset \neq S \subseteq V(G),$$

then G has a tree with leaf distance at least d .

Since the leaf distance of a tree is at least 3 if and only if its maximum leaf degree is one, from Theorem 4.6.2 with $m = 1$, we see that Conjecture 4.6.1 holds for $d = 3$. Recently, Kaneko, Kano and Suzuki [270] confirmed the conjecture for $d = 4$. But, for $d \geq 5$, the conjecture is still open.

A related problem, which considers a tree with a restricted degree on portion of the tree, was studied and a characterization was obtained.

Theorem 4.6.3 (Kaneko and Yoshimoto [272]; Frank and Gyárfás [216]). *Let S be an independent set of a graph G and $f : S \rightarrow \{2, 3, \dots\}$. Then G has a tree T such that $d_T(x) \geq f(x)$ for all $x \in S$ if and only if*

$$\left| \bigcup_{x \in X} N_G(x) \right| \geq f(X) - |X| + 1 \quad \text{for all } \emptyset \neq X \subseteq S.$$

Next we turn our attention to k -trees. Ore (1960) showed that if $\sigma_2(G) \geq |V(G)| - 1$, then G has a 2-tree (i.e., Hamilton path). A generalization to k -trees was obtained by Win (1975).

Theorem 4.6.4 (Win [584]). *Let G be a graph of order n . If $\sigma_k(G) \geq n - 1$, then G has a k -tree.*

The above theorem has been generalized to other conditions, such as combination of degree sum and neighborhood union. On the other hand, using Chvátal and Erdős condition, Neumann-Lara and Rivera-Campo [443] obtained a general result about trees with a bounded degree. We start with a few terms and a lemma.

Let T be a tree of G . We denote by $V_1(T)$ the set of leaves in T (i.e., vertices of degree one). For any two vertices u and v , T_{uv} denotes the unique path in T joining u and v . Choose any vertex of T as its root vertex, an *out-directed tree* $\vec{T}$ is a rooted tree in which all the edges are directed away from the root. Whenever $\vec{T}$ is an out-directed tree with vertex set $V(\vec{T}) = V(T)$ and arc set $A(\vec{T})$, given a subset U of $V(\vec{T})$, we shall denote by $N^+(U)$ the set of vertices $w \in V(T)$ for which there is an arc $uw \in A(\vec{T})$ for some $u \in U$; $N^-(U)$ is defined similarly. For any $u \in V(T)$, the set $N^-(\{u\})$ consists of a unique vertex which will be denoted by u^- .

Lemma 4.6.1 ([443]). *Let $\vec{T}$ be an out-directed tree, and let R and B be disjoint subsets of $V(T)$ such that $(R \cup B) \cap V_1(T) = \emptyset$ and $N^+(R \cup B) \cap R = \emptyset$. For each $b \in B$ let x_b be any fixed vertex in $N^+(b)$ and let $X(B) = \{x_b \mid b \in B\}$. There exists an injective function $\phi : N^+(R) \cup (N^+(B) - X(B)) \rightarrow V_1(T) \cup N^-(R)$ which satisfies:*

- (1) *if $u \in N^+(R) \cup (N^+(B) - X(B))$, then $T_{u\phi(u)}$ is u to $\phi(u)$ directed in $\vec{T}$;*
- (2) *$T_{u\phi(u)}$ and $T_{v\phi(v)}$ are vertex disjoint, whenever $u \neq v$;*
- (3) *if $u \neq v$, then at least one of u^- and v^- is a vertex of $T_{\phi(u)\phi(v)}$;*
- (4) *the range of ϕ is an independent set in T .*

Proof. Let $Y = \{ba \in A(\vec{T}) \mid b \in B, a \neq x_b\}$. Clearly, each component of the directed graph $\vec{H} = (\vec{T} - R) - Y$ is an out-directed tree; moreover each $u \in N^+(R) \cup (N^+(B) - X(B))$ is the root of the component $\vec{H}_u$ of $\vec{H}$ that contains u , and each leaf of $\vec{H}_u$ is either a leaf of $\vec{T}$ or is in $N^-(R)$. Therefore, we can define ϕ by letting $\phi(u)$ be any leaf of $\vec{H}_u$; Conditions (1) and (2) are satisfied by the construction.

Let r be the root of $\vec{T}$, and let u and v be two distinct vertices in $N^+(R) \cup (N^+(B) - X(B))$. The paths $T_{u\phi(u)}$ and $T_{v\phi(v)}$ are contained in $T_{r\phi(u)}$ and $T_{r\phi(v)}$, respectively. Let $w \in V(T_{r\phi(u)}) \cap V(T_{r\phi(v)})$ be a vertex such that $T_{\phi(u)\phi(v)} = T_{w\phi(u)} \cup T_{w\phi(v)}$. By Condition (2), $T_{u\phi(u)}$ and $T_{v\phi(v)}$ are vertex disjoint; hence if $w = \phi(u)$, then $v^- \in V(T_{\phi(u)\phi(v)})$; if $w = \phi(v)$, then $u^- \in V(T_{\phi(u)\phi(v)})$; if $\phi(u) \neq w \neq \phi(v)$, then $u^- \in V(T_{w\phi(u)})$, $v^- \in V(T_{w\phi(v)})$.

Finally, suppose that the arc $\phi(u)\phi(v)$ is in $A(\vec{T})$. Since $T_{u\phi(u)}$ and $T_{v\phi(v)}$ are vertex disjoint, then $v = \phi(v)$ and thus $v^- = \phi(u)$. By the construction, v^- is in $R \cup B$ and $\phi(u)$ is in $V_1(T) \cup N^-(R)$ but $(R \cup B) \cap (V_1(T) \cup N^-(R)) = \emptyset$. Therefore, the range of ϕ is an independent set in T or (4) is satisfied. $\square$

Theorem 4.6.5 (Neumann-Lara and Rivera-Campo [443]). *Let G be a k -connected graph, and let s and c be integers with $s \geq 3$, $0 \leq c \leq k$. If $\alpha(G) \leq k(s-1) + c + 1$, then G has an $(s+1)$ -tree T with at most c vertices of degree $s+1$.*

Proof. A subgraph H of G is called an $(s+1, c)$ -subtree if H is an $(s+1)$ -tree (may not be spanning) with at most c vertices of degree $s+1$.

Let T be an $(s+1, c)$ -subtree of G with the maximum possible number of vertices. If T is not spanning, we choose $w \in V(G) - V(T)$. Let $P = \{\pi_1, \pi_2, \dots, \pi_l\}$ be a maximum collection of internally disjoint w - T paths in G . For each i , let r_i be the unique vertex of π_i in T . Assume $d_T(r_1) \leq d_T(r_2) \leq \dots \leq d_T(r_l)$. By the choice of T , we have $s \leq d_T(r_1) \leq d_T(r_l) \leq s+1$; and from Menger Theorem, we see that $l \geq k$.

Let $R = \{r_1, r_2, \dots, r_l\}$ and t be the number of vertices in R with degree s in T . Let $B = \{b_1, b_2, \dots, b_m\}$ be the set of vertices in $V(T) - R$ having degree $s+1$ in T . If $t > 0$, in particular, $d_T(r_1) = s$, then $T \cup \pi_1$ is a subtree of G having degrees at most $s+1$. Since T is a maximum $(s+1, c)$ -subtree of G , then it must contain exactly c vertices of degree $s+1$; therefore $m = c - l + t$. When $t = 0$, then $d_T(r_i) = s+1$ for $i = 1, 2, \dots, l$; hence $c \geq l$, but $l \geq k \geq c$; therefore, in this case $c = l = k$, $B = \emptyset$ and $m = 0$.

Consider the out-directed tree $\vec{T}$ with root r_1 . Clearly, R and B are disjoint and $(R \cup B) \cap V_1(T) = \emptyset$. If $r_i r_j$ is an arc of $\vec{T}$, then $T' = (T - r_i r_j) \cup \pi_i \cup \pi_j$ is a $(s+1, c)$ -subtree of G containing more vertices than T . If $b_i r_j$ is an arc of $\vec{T}$, then $T' = (T - b_i r_j) \cup \pi_i \cup \pi_j$ is an $(s+1, c)$ -subtree of G larger than T . Therefore, we also have $N^+(R \cup B) \cap R = \emptyset$.

Choose $X(B)$ and ϕ as in Lemma 4.6.1 and let W be the range of ϕ . Since ϕ is a 1-1 mapping, then

$$\begin{aligned}
 |W| &= |N^+(R) \cup (N^+(B) - X(B))| \\
 &= |N^+(R)| + |(N^+(B) - X(B))| \\
 &= [1 + t(s-1) + (l-t)s] + [(c-l+t)(s-1)] \\
 &= 1 + l(s-1) + (l-t) + (c-l+t)(s-1) \\
 &\geq 1 + l(s-1) + c \\
 &\geq 1 + k(s-1) + c.
 \end{aligned}$$

Since $\alpha(G) \leq 1 + k(s-1) + c$, then $W \cup \{w\}$ can not be an independent set in G . Hence, there is an edge $xy \in E(G)$ with both end-vertices in $W \cup \{w\}$. In addition, by the choice of P , any vertex in T adjacent to w in G must lie in R . But $R \cap W \subset R \cap (V_1(T) \cup N^-(R)) = \emptyset$, so both x and y must be in W . By Lemma 4.6.1, W is independent in T , then $xy \in E(G) - E(T)$.

Let $u, v \in N^+(R) \cup (N^+(B) - X(B))$ such that $\phi(u) = x$ and $\phi(v) = y$. By Condition (3) in Lemma 4.6.1, we can choose a vertex $p(x, y) \in \{u^-, v^-\} \cap V(T_{xy})$.

We prove that T may be transformed into a tree T' containing all vertices of T and the vertex w . The tree T' is formed by deleting some edges of T , breaking it into several components and then joining these components by adding the edge xy and some of the w - T paths in P .

This procedure is done in such a way that when an edge is added to a vertex λ of T , then either λ is a leaf of T (in this case, another edge incident with λ is deleted) or λ is r_1 . In the latter case, then $d_T(r_1) = s$, $d_T(p(x, y)) = s + 1$ and an edge of T incident with $p(x, y)$ has been deleted. Hence the maximum degree and the number of vertices with degree $s + 1$ is unchanged from T to T' .

We consider several cases as follows.

Let z be a vertex in $V(T_{xy})$ adjacent to $p(x, y)$ and, for each $\alpha \in N^-(R)$, let $i(\alpha)$ be an integer such that $\alpha r_{i(\alpha)}$ is an arc of $\vec{T}$.

Case 1. $x \in V_1(T)$ and $y \in V_1(T)$.

If $p(x, y) \in R$, say $p(x, y) = r_i$, then

$$T' = ((T \cup xy) - r_i z) \cup \pi_i \text{ (see Fig. 4.7 (a)).}$$

Otherwise, $p(x, y) = b_i$ for some i and then

$$T' = ((T \cup xy) - b_i z) \cup \pi_1 \text{ (see Fig. 4.7 (b)).}$$

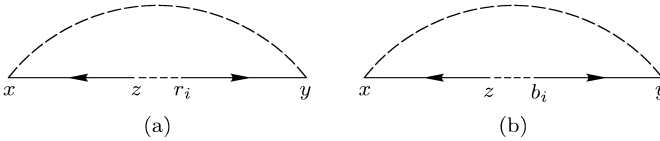


Fig. 4.7 The constructions of T' in Case 1.

Case 2. $x \in V_1(T)$ and $y \in N^-(R)$.

If $r_{i(y)} \in V(T_{xy})$, then

$$T' = ((T \cup xy) - yr_{i(y)}) \cup \pi_{i(y)} \text{ (see Fig. 4.8 (a)).}$$

Otherwise, $r_{i(y)} \notin V(T_{xy})$, in which case when $p(x, y) \in R$, say $p(x, y) = r_j$, then

$$T' = (((T \cup xy) - r_j z) - yr_{i(y)}) \cup \pi_j \cup \pi_{i(y)} \text{ (see Fig. 4.8 (b)),}$$

and when $p(x, y) = b_j$ for some j , then

$$T' = (((T \cup xy) - b_j z) - yr_{i(y)}) \cup \pi_1 \cup \pi_{i(y)} \text{ (see Fig. 4.8 (c)).}$$

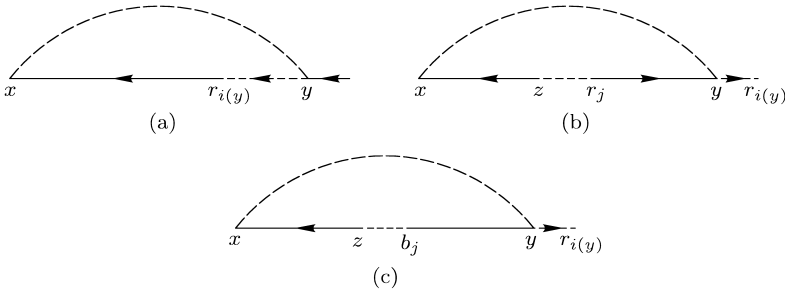


Fig. 4.8 The constructions of T' in Case 2.

Case 3. $x \in N^-(R)$ and $y \in V_1(T)$.

In this case, T' is constructed by interchanging x and y in Case 2.

Case 4. $x \in N^-(R)$ and $y \in N^-(R)$.

If either $r_{i(x)}$ or $r_{i(y)}$ lies in $V(T_{xy})$, then

$$T' = (((T \cup xy) - xr_{i(x)}) - yr_{i(y)}) \cup \pi_{i(x)} \cup \pi_{i(y)}.$$

Otherwise, $r_{i(x)} \notin V(T_{xy})$, $r_{i(y)} \notin V(T_{xy})$, in which case, when $p(x, y) \in R$, say $p(x, y) = r_h$, then

$$T' = (((((T \cup xy) - r_h z) - xr_{i(x)}) - yr_{i(y)}) \cup \pi_h \cup \pi_{i(x)} \cup \pi_{i(y)},$$

and when $p(x, y) = b_h \in B$, then

$$T' = (((((T \cup xy) - b_h z) - xr_{i(x)}) - yr_{i(y)}) \cup \pi_1 \cup \pi_{i(x)} \cup \pi_{i(y)}.$$

Cases 1 to 4 cover all possibilities and in each case T' is an $(s+1, c)$ -subtree of G larger than T which contradicts the choice of T . Therefore, T is a spanning $(s+1, c)$ -subtree of G . $\square$

Applying the same arguments but with less complexity than the proof of Theorem 4.6.5, Neumann-Lara and Rivera-Campo [443] also proved the following weaker but simpler and more comprehensible version of Theorem 4.6.5. Note that this result is clearly a generalization of Corollary 4.4.1.

Theorem 4.6.6 (Neumann-Lara and Rivera-Campo [443]). *Let G be a k -connected graph with independent number $\alpha(G)$. If $\alpha(G) \leq ks + 1$ for some positive integer s , then G has an $(s+1)$ -tree.*

More recently, Matsuda and Matsumura (2006) used degree sum condition to insure existence of k -trees with specified leaves.

Theorem 4.6.7 (Matsuda and Matsumura [429]). *Let G be an $(r+1)$ -connected graph of order n and S any subset of $V(G)$ with $|S| = r$. For an integer k with $k \geq 2$ and $0 \leq r \leq k$, if*

$$\sigma_k(G) \geq n + (k-1)r - 1,$$

then G has a k -tree such that every vertex of S is a leaf.

In Chapter 3, we have discussed the relationship between toughness and factors. In fact, toughness plays an important role in the study of factor theory. The following theorem on k -tree using toughness as a sufficient condition was given by Win [587] and the short proof here comes from Ellingham and Zha [181]. In this proof, a S -bridge of a vertex-set S in a graph G is either an edge joining two vertices of S and its ends, or a component of $G - S$ together with the edges joining it to S and their ends. For example, in Fig. 4.9, $S = \{a, b, c, d\}$ and we have five S -bridges, two of which are single edges.

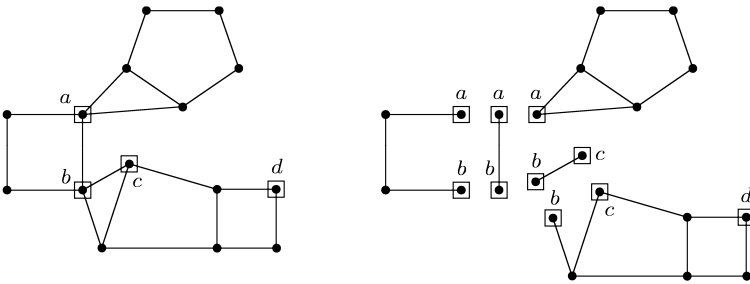


Fig. 4.9 An illustration of S -bridges.

A toughness condition for k -tree was obtained by Win (1989). Essentially, he proved that for $k \geq 3$, a $\frac{1}{k-2}$ -tough graph has a k -tree.

Theorem 4.6.8 (Win [587]). *Suppose that $k \geq 2$ and G is a connected graph such that $\omega(G - S) \leq (k-2)|S| + 2$ for all $S \subseteq V(G)$. Then G has a k -tree.*

Proof. Let H be an induced subgraph of G such that H has a k -tree with the maximum order. Suppose that $H \neq G$. Let A_0 be the set of vertices of H that have degree k in every k -tree of H . By maximality of H , every edge from $V(G) - V(H)$ to $V(H)$ is incident with a vertex of A_0 . Since G is connected, A_0 is, therefore, nonempty.

Fix some k -tree T of H . Given $S \subseteq V(H)$, $\Gamma(S)$ be the set of k -tree T' of H such that the S -bridges of S in T' have the same vertex sets as the S -bridges of S in T . Suppose that $T_1, T_2 \in \Gamma(S)$. Let B_1 be an S -bridge in T_1 and B_2 the corresponding S -bridge in T_2 . Both B_1 and B_2 contain exactly

one edge incident with each vertex of $V(B_1) \cap S = V(B_2) \cap S$. Therefore, we may replace B_1 in T_1 by B_2 to obtain a new tree $T_3 \in \Gamma(S)$.

Let A_1 denote the set of vertices of H that have degree k in every element of $\Gamma(A_0)$; clearly, $A_0 \subseteq A_1$. Suppose that there is some $uv \in E(H)$ joining vertices in two different components of $T - A_0$. Then u and v are in different components of $T' - A_0$ for any $T' \in \Gamma(A_0)$. Suppose that $u \notin A_1$ and $v \notin A_1$. Then there are $T_1, T_2 \in \Gamma(A_0)$ such that $d_{T_1}(u) < k$ and $d_{T_2}(v) < k$. Replace the A_0 -bridge in T_1 containing v with the A_0 -bridge in T_2 containing v to obtain $T_3 \in \Gamma(A_0)$, in which both u and v have degree less than k . There is at least one $w \in A_0$ on the path $T_3(u, v)$ between u and v in T_3 . By adding uv to T_3 and deleting one of the edges of $T_3(u, v)$ incident with w , we obtain a k -tree of H in which w has degree less than k , contradicting to $w \in A_0$. Therefore, either $u \in A_1$ or $v \in A_1$ or both, i.e., every edge of H joining two components of $T - A_0$ has at least one end in A_1 .

Now let A_2 denote the set of vertices of H that have degree k in every element $\Gamma(A_1)$; clearly, $A_1 \subseteq A_2$. Any $uv \in E(H)$ joining two components of $T - A_1$ must join two vertices within a single component of $T - A_0$. If neither u nor v is in A_2 , then similarly to above, we can find a tree $T_3 \in \Gamma(A_1)$ in which both u and v have degree less than k , and there is a vertex w of $A_1 - A_0$ on the path $T_3(u, v)$; but adding uv and deleting an edge of $T_3(u, v)$ incident with w yields a tree contradicting to $w \in A_1$. Thus, every edge of H joining two components of $T - A_1$ has at least one end in A_2 .

We may continue in this way in general to define A_{j+1} to be the set of vertices of H with degree k in every element of $\Gamma(A_j)$, and deducing that every edge of H joining two components of $T - A_j$ has both ends in A_{j+1} . Thus, we have $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_j \subset \cdots \subseteq D_k(T)$, where $D_k(T)$ is the set of vertices of degree k in T . Since $D_k(T)$ is finite, we must have $A_m = A_{m+1}$ for some m . Then we have a set A_m of vertices of degree k in T such that every edge of H joining two components of $T - A_m$ has at least one end in A_m , i.e., there are no such edges.

Thus, we have a set $S = A_m$ of vertices of degree k in H with the property that every edge between $V(H)$ and $V(G) - V(H)$ has an end in S , and the vertex sets of the components of $H - S$ and of $T - S$ are the same. Then, as in [587] $\omega(H - S) = \omega(T - S) \geq (k - 2)|S| + 2$, because T is a tree in which all vertices of S have degree k . Since $H \neq G$, we get that $\omega(G - S) > (k - 2)|S| + 2$, a contradiction. Thus $G = H$ has a k -tree. $\square$

Given a factor F in a graph G , an F -forest is a subgraph H of G such that every component of F is either contained in or vertex-disjoint from H , and such that every edge of $E(H) - E(F)$ is a cut edge of H . A connected F -forest is an F -tree. Given a nonnegative integer-valued function h on $V(G)$, we define an (F, h) -forest to be an F -forest H with $V(H) = V(G)$ and $d_H(v) \leq d_F(v) + h(v)$ for all $v \in V(G)$.

Ellingham, Nam and Voss (2002) obtained a structural characterization of connected (F, h) -forest and a polynomial-time algorithm to find a “barrier” in

case of connected (F, h) -forest does not exist. Using the characterization, they were able to obtain a set of sufficient conditions for connected (F, h) -forests.

Theorem 4.6.9 (Ellingham, Nam and Voss [180]). *Let G be a connected graph, and h be a nonnegative integer-valued function on $V(G)$. Assume that G has a factor F in which each component has at least α vertices. Then G has a connected (F, h) -forest if for every nonempty $S \subseteq V(G)$ at least one of the following holds:*

- (a) $\omega(G - S) < h(S) - 2|S| + 3$; or
- (b) $\alpha \geq 2$ and $\omega(G - S) < h(S) - |S| + 3$; or
- (c) $\omega(G - S) < \lceil \frac{1}{2}h(S) - |S|/\alpha + 2 \rceil$; or
- (d) $\alpha k \geq 2$, and $\omega(G - S) < \lceil \frac{1}{2}(\frac{\alpha k - 2}{\alpha k - 1})h(S) + \frac{2\alpha k - 1}{\alpha k - 1} \rceil$, where $k = \min\{h(v) \mid v \in S\} \geq 1$.

This theorem appears somewhat inelegant; however, the conditions given do seem to be extremely useful, as several previous known theorems, including Theorems 4.5.4 and 4.6.8, can be derived from it using different counting arguments. We present a few other corollaries below.

Corollary 4.6.1. *Let G be a connected graph, and k an integer with $k \geq 1$. Let F be a factor in which each component has at least α vertices. Then G has a connected (F, k) -forest, if, for every $S \subseteq V(G)$, we have*

$$\omega(G - S) < \begin{cases} (k - 2)|S| + 3 & \text{if } k \geq 2, \alpha = 1; \\ (k - 1)|S| + 3 & \text{if } k \geq 2, \alpha \geq 2; \\ \frac{1}{2} \left(\frac{\alpha - 2}{\alpha - 1} \right) |S| + \frac{2\alpha - 1}{\alpha - 1} & \text{if } k = 1, \alpha \geq 2. \end{cases}$$

Corollary 4.6.2. *Let k be an integer greater than two, G be a graph with $t(G) \geq \frac{1}{k-2}$, and f and g be nonnegative integer-valued functions on $V(G)$ with $f \geq g$. If G has a (g, f) -factor F , then G has a connected $(g, f + k)$ -factor containing F .*

Corollary 4.6.3. *Let G be a simple graph, and f and g be positive integer-valued functions on $V(G)$ with $1 < g \leq f$. Let $g_{\min} = \min\{g(v) \mid v \in V(G)\}$. If toughness $t(G) \geq \frac{2g_{\min}}{g_{\min} - 1}$, and G has a (g, f) -factor F , then G has a connected $(g, f + 1)$ -factor containing F .*

Applying Theorem 4.6.9 and with high connectivity, Ellingham, Nam and Voss (2002) proved that there exists a tree with relatively low degree (i.e., sparse tree).

Theorem 4.6.10 (Ellingham, Nam and Voss [180]). *If G is an m -edge-connected graph with $m \geq 1$, then G has a tree T such that*

$$d_T(v) \leq 2 + \lceil \frac{d_G(v)}{m} \rceil$$

for every vertex $v \in V(G)$.

Recently, spanning tree problems have attracted more attention of many researchers. In this section, we summary only a few results in this field. More generalizations and progresses in this direction can be found from Kano and Kyaw's survey paper [285].

Chapter 5

Elementary Graphs and Decomposition Theory

There are many graphs that are 1-factor rich, i.e., containing many 1-factors. For example, n -factor-critical graphs, k -extendable graphs and various matching-extendable graphs. However, we concentrate on three families of such graphs, *elementary graphs*, *1-extendable graphs* and *bicritical graphs*, where each family is a subset of later family. There are two reasons for investing time on these families: firstly, we want to have better understanding of their structures and properties, including many equivalent characterizations; secondly, they play important roles in the study of graphs with perfect matchings, or served as a building block in decompositions of graphs.

In the first two sections of this chapter, we investigate their properties and criteria of various matching extendable graphs. Furthermore, to develop decomposition theories for various families of graphs. In the later sections, we focus on minimal graphs, bricks and brick decomposition. In particular, we survey the recent developments in the area of optimal ear decomposition.

Matching theory is a well-studied and matured subject in graph theory, one indication for its maturity is that there are many criteria and structural results for elementary graphs, factor-critical graphs, bicritical graphs, etc. We present several decompositions: brick decomposition for elementary graphs (Section 5.1), ear decomposition (Section 5.2), end decomposition (Section 5.3), optimal ear decomposition (Section 5.4).

In the excellent book by Lovász and Plummer [412], there were good treatment in some of these topics and we follow the same line to deal with elementary graphs, in particular in the first two sections. The results in this chapter, when not referenced on the spot, are due to Lovász or/and Plummer [401; 404; 411; 412].

5.1 Elementary graphs and 1-extendable graphs

In this section, we give a deeper treatment on the structure of graphs with perfect matchings. Many techniques present here are very powerful and deserve reader's attention. Firstly, we review a few terminologies.

If an edge lies in a 1-factor of a graph, then we call such an edge *allowed* and an edge not containing in any 1-factor is *forbidden*. For an edge e and a 1-factor M , if $e \notin M$, we say M *avoiding* edge e .

A connected graph G with a 1-factor is *elementary* if its allowed edges form a connected (spanning) subgraph. If every edge of G is allowed, then G is called *1-extendable*. For any two distinct vertices $x, y \in V(G)$, if $G - \{x, y\}$ has a 1-factor, then G is a *bicritical graph*.

It is not hard to see that every bicritical graph is 1-extendable, and any 1-extendable connected graph is elementary. That is,

$$(\text{bicritical graphs}) \Rightarrow (1\text{-extendable graphs}) \Rightarrow (\text{elementary graphs}). \quad (5.1)$$

A digraph D is *strongly connected* if there exists a directed path from x to y and a directed path from y to x for any distinct vertices $x, y \in V(D)$. Next we introduce an auxiliary digraph to associate with a bipartite graph. Let G be a bipartite graph with bipartition (X, Y) and with a 1-factor M . Orient all edges of G toward X and contract the edges of M , the resulting digraph is denoted by $D = D(G, M)$. It is not hard to see that a bipartite graph G is 1-extendable if and only if $D(G, M)$ is strongly connected [412]. Moreover, if G has a unique perfect matching, then $D(G, M)$ is acyclic.

We collect several characterizations of elementary graphs in the next theorem. There are more criteria for this family of graphs (such as odd ear decomposition) given in the future sections.

Theorem 5.1.1 (Hetyei (1964)). *Suppose $G = (X, Y)$ is a bipartite graph. Then the following are equivalent:*

- (a) G is elementary;
- (b) G has exactly two minimum vertex covers, namely X and Y ;
- (c) $|X| = |Y|$ and for every non-empty proper subset S of X , $|N(S)| \geq |S| + 1$;
- (d) $G = K_2$, or $|V(G)| \geq 4$ and for any $x \in X, y \in Y$, $G - \{x, y\}$ has a 1-factor;
- (e) G is connected and every edge of G is allowed (i.e., G is 1-extendable);
- (f) For any 1-factor M of G , $D(G, M)$ is strongly connected.

Proof. (a) $\Rightarrow$ (b). Suppose that G has a minimum vertex cover K such that $K_X = K \cap X \neq \emptyset$ and $K_Y = K \cap Y \neq \emptyset$. Suppose $G[K]$ contains an allowed edge xy . Let M be a perfect matching in G containing xy . Then M matches $X - K_X$ into K_Y and $Y - K_Y$ into K_X . Hence $|X - K_X| < |K_Y|$. But X is

a cover and $|X| = |K_X| + |X - K_X| < |K_X| + |K_Y| = |K|$, contradicting to the minimality of K . Thus all edges in $G[K]$ are forbidden. But removal of all edges in $G[K]$ necessarily disconnects G and thus G is not elementary.

(b) $\Rightarrow$ (c). That $|X| = |Y|$ is immediate. Suppose there is a non-empty proper subset S in X , i.e., $S \subseteq X$ and $\emptyset \neq S \neq X$, such that $|N(S)| \leq |S|$. Now $(X - S) \cup N(S)$ covers G and $|(X - S) \cup N(S)| = |X - S| + |N(S)| \leq |X - S| + |S| = |X|$ and hence $(X - S) \cup N(S)$ is a minimum vertex cover of G . But $X - S \neq \emptyset$ and thus by assumption $N(S) = \emptyset$, that is, S is a set of isolated vertices. But then $X - S$ covers G and $|X - S| < |X|$, a contradiction.

(c) $\Rightarrow$ (d). Suppose $G \neq K_2$ and thus $|V(G)| \geq 4$. Let $x \in X$, $y \in Y$ and $H = G - \{x, y\}$. For any subset $\emptyset \neq S \subseteq X - \{x\}$, S is a non-empty proper subset of X and thus $|N_G(S)| \geq |S| + 1$. Therefore, $|N_H(S)| \geq |N_G(S)| - 1 \geq |S|$ and by Hall's Theorem H has a 1-factor.

(d) $\Rightarrow$ (e). If $G = K_2$ we are done. So suppose $|V(G)| \geq 4$ and G is not connected. Let C be a component of G such that $|V(C) \cap X| \leq |V(C) \cap Y|$. Let $x \in V(C) \cap X$ and $y \in Y - V(C)$. Then $G - \{x, y\}$ has no perfect matchings, as $G - \{x, y\}$ is a imbalance bipartite graph. Thus G is connected. So it is immediate that each edge of G is allowed.

(e) $\Rightarrow$ (a). Trivial.

(d) $\Rightarrow$ (f). Let $X = \{x_1, x_2, \dots, x_r\}$ and $Y = \{y_1, y_2, \dots, y_r\}$. Let $M = \{x_1y_1, x_2y_2, \dots, x_ry_r\}$ be a 1-factor of G . Denote v_i the vertex obtained by contracting x_iy_i . For any two distinct vertices u_i and u_j , $G - \{x_i, y_j\}$ has a 1-factor M' , then $M \triangle M'$ has an alternating M - M' path starting with y_i and ending with x_j . Clearly, this alternating path corresponds to a direct path from u_i to u_j in $D(G, M)$. Similarly, $G - \{y_i, x_j\}$ has a 1-factor M'' , $M \triangle M''$ has an alternating M - M'' path starting with y_j and ending with x_i , then it corresponds to a direct path from u_j to u_i in $G(D, M)$.

(f) $\Rightarrow$ (d). Trivial. □

We can see from Theorem 5.1.1 that a bipartite graph is elementary if and only if it is 1-extendable. For 1-extendable general graphs, a characterization was obtained by Little, Grant and Holton (1975).

Theorem 5.1.2 (Little, Grant and Holton [349]). *Let G be a connected graph. Then G is a 1-extendable graph if and only if*

$$o(G - S) \leq |S| \quad \text{for all } S \subset V(G)$$

and

$$o(G - S) = |S| \quad \text{only if } G[S] \text{ is an independent set.}$$

Proof. To see the necessity, let S be any subset of $V(G)$, since G contains 1-factors, $o(G - S) \leq |S|$ by Tutte's 1-Factor Theorem. Assume that $o(G - S) = |S|$ for some S and $G[S]$ contains an edge, say $e = xy$. Since G is 1-extendable, there exists a 1-factor F containing e . However, there exists an edge in F joining each odd component C to a vertex in $S - \{x, y\}$, and hence

$$o(G - S) \leq e_F(G - S, S - \{x, y\}) \leq |S - \{x, y\}| = |S| - 2,$$

a contradiction.

Conversely, let $e = xy$ be any edge of G . To see that G is 1-extendable, we need only to show that there exists a 1-factor F containing e or $G - \{x, y\}$ has a 1-factor. Let $G' = G - \{x, y\}$ and S' be any subset of $V(G')$. Since $G[S' \cup \{x, y\}]$ contains an edge e , we have $o(G - (S' \cup \{x, y\})) < |S' \cup \{x, y\}|$ or $o((G - \{x, y\}) - S') \leq |S'|$ by parity. Thus, by Tutte's 1-Factor Theorem, $G - \{x, y\}$ has a 1-factor. $\square$

An edge e of a connected graph G is called an *odd-bridge* if e is a bridge of G and $G - e$ consists of two odd components. The characterization of a graph with a 1-factor avoiding any given edge was obtained by Chen (1995).

Theorem 5.1.3 (Chen [136]). *Let G be a connected graph. Then for every edge $e \in E(G)$, G has a 1-factor avoiding e if and only if*

$$o(G - S) \leq |S| - \varepsilon_2 \text{ for all } S \subseteq V(G), \quad (5.2)$$

where $\varepsilon_2 = 2$ if $G - S$ has a component containing an odd-bridge; otherwise $\varepsilon_2 = 0$.

Proof. Let $e = xy \in E(G)$ and assume that there exists a 1-factor F avoiding e . Since G has 1-factors, by Tutte's 1-Factor Theorem, $o(G - S) \leq |S|$ for any $S \subseteq V(G)$. Assume that $G - S$ has an (even) component C_0 containing an odd-bridge e . Thus $G - S - e$ has $o(G - S) + 2$ components. Note that each odd component is joined to S by at least one edge of F , so we have

$$o(G - S) + 2 = o(G - S - e) \leq e_F(G - S - e, S) \leq |S|,$$

or (5.2) holds.

Conversely, to see that there exists a 1-factor of G avoiding any given edge e , we need to show that $G - e$ has a 1-factor. Let $G' = G - e$ and $S' \subseteq V(G')$. If e is an odd-bridge of an even component of $G - S'$, then $o(G - e - S') = o(G - S') + 2$; otherwise, it is not hard to see that $o(G - e - S') = o(G - S')$. Therefore

$$o(G' - S') = o(G - S') + \varepsilon_2 \leq |S'|.$$

From Tutte's 1-Factor Theorem, G' has a 1-factor. That is, G has a 1-factor avoiding e . $\square$

Recall that $def(G)$ is the number of vertices missed by a maximum matching of G . Berge's Formula implies that $def(G) = \max\{o(G - S) - |S| \mid S \subseteq V(G)\}$.

Recall that a vertex subset S of G is called a *barrier* in G if $o(G - S) = |S| + def(G)$. So the barrier is a vertex subset that produce the maximum

value on the right hand side of Berge's Formula. The empty set may or may not be a barrier in general. However, if G has a 1-factor, $\emptyset$ is a barrier. By the Gallai-Edmonds Structure Theorem, the set $A(G)$ is a barrier but may not be unique.

The concept of barriers is a useful tool to study the maximum matchings in graphs. So we collect some of the basic properties of barriers in the next theorem.

Theorem 5.1.4. *Let G be a graph and S be a barrier in G . Then*

- (a) $S \subseteq A(G) \cup C(G)$;
- (b) if $S' \subseteq S$, then $S - S'$ is a barrier in $G - S'$. In particular, if $x \in S$, then $S - \{x\}$ is a barrier in $G - \{x\}$;
- (c) $S \cup A(G - S)$ is a barrier in G ;
- (d) if G has a 1-factor, then an edge e of G is allowed if and only if no barrier contains both ends of e ;
- (e) a barrier S of G is maximal if and only if all components of $G - S$ are factor-critical;
- (f) intersection of two maximal barriers is a barrier;
- (g) $A(G)$ is the intersection of all maximal barriers in G .

There is another important tool in the study of graphs with a 1-factor—canonical partition, which is useful for investigating the location of allowed edges or ears (see next section for more details). Much research is carried out on this subject and reader can find many results from [412]. The *canonical partition* $\mathcal{P}(G)$ is defined to be a collection of vertex subsets $\{S_1, S_2, \dots, S_k\}$ of G , where $S_1, S_2, \dots, S_k$ are all the maximal barriers in G (see Fig. 5.1). There is an alternative definition for canonical partition by using an equivalence relation. Define a binary relation $\sim$ on $V(G)$ where $u \sim v$ if $u = v$ or $G - \{u, v\}$ has no 1-factors. It can be shown that the relation $\sim$ is an equivalence relation on V and the equivalence classes are precisely the (inclusionwise) maximal barriers in G . For example, the classes of the canonical partition of a bicritical graph are all singletons. Another trivial example is bipartite graphs, in this case, $\mathcal{P}(G)$ consists of two classes, being just the two color classes.

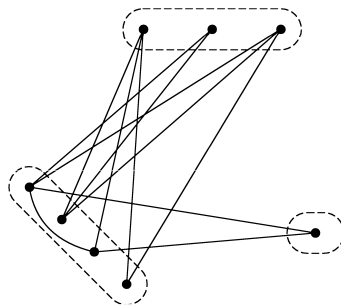


Fig. 5.1 Canonical partition $\mathcal{P}(G)$ of a graph G with order 8.

In the next theorem, we use barriers, Gallai-Edmonds Structure and the canonical partition to give criteria for elementary graphs.

Theorem 5.1.5. *Let G be a connected graph, then the following statements are equivalent:*

- (a) G is elementary;
- (b) $o(G - S) \leq |S|$ for all $S \subseteq V(G)$, and if $o(G - S) = |S|$ holds for some $S \neq \emptyset$ (i.e., S is a non-empty barrier in G), then $G - S$ has no even components;
- (c) G has a 1-factor and for all $x \in V(G)$, $C(G - x) = \emptyset$;
- (d) $\mathcal{P}(G) = \{S_1, S_2, \dots, S_k\}$ is a partition of $V(G)$.

Part (b) is a “Tutte-type” characterization for the elementary graphs. Note the similarity of this result with the criterion of 1-extendable graphs (i.e., Theorem 5.1.2).

There are more equivalent conditions for elementary graphs. For example, at the end of this section, we present “bicritical decomposition” and in the next section, we see ear decomposition for elementary graphs.

The next theorem summarizes some important properties of the partition $\mathcal{P}(G)$.

Theorem 5.1.6. *Let $\mathcal{P}(G) = \{S_1, S_2, \dots, S_k\}$ be the canonical partition of an elementary graph G . Then*

- (a) if x and y are distinct vertices of G , then $G - \{x, y\}$ has a 1-factor if and only if x and y lie in different classes of $\mathcal{P}(G)$;
- (b) an edge $e = xy \in E(G)$ is allowed in G if and only if x and y lie in different classes of $\mathcal{P}(G)$;
- (c) if $f = uv \notin E(G)$ and $V(f) \subseteq V(G)$, then f is allowed in $G \cup \{f\}$ if and only if u and v lie in different classes of $\mathcal{P}(G)$. Moreover, $\mathcal{P}(G \cup \{f\})$ is a refinement of $\mathcal{P}(G)$;
- (d) if $\emptyset \neq S \subseteq S_i \in \mathcal{P}(G)$, then $A(G - S) = S_i - S$ and $C(G - S) = \emptyset$;
- (e) let $S \subseteq V(G)$. Then $S \in \mathcal{P}(G)$ if and only if $G - S$ has exactly $|S|$ components and each of which is factor-critical.

Finally, we present the characterizations of bicritical graphs. The first one is “Tutte-type” condition and the second is in terms of canonical partition $\mathcal{P}(G)$.

Theorem 5.1.7. *For a graph G , the following are equivalent statements.*

- (a) G is bicritical;
- (b) If $S \subseteq V(G)$ and $|S| \geq 2$, then $o(G - S) \leq |S| - 2$;
- (c) G is elementary and all classes of $\mathcal{P}(G)$ are singletons.

Proof. (a) $\Rightarrow$ (c). Clearly, G is elementary. Suppose G is bicritical and at least one element of $\mathcal{P}(G) = \{S_1, S_2, \dots, S_k\}$ is not singleton, say $|S_1| \geq 2$. Let $x, y \in S_1$. Since S_1 is a barrier, we have $o(G - S_1) = |S_1| + \text{def}(G) = |S_1|$, and so $o((G - \{x, y\}) - (S_1 - \{x, y\})) = o(G - S_1) = |S_1 - \{x, y\}| + 2$. Thus $\text{def}(G - \{x, y\}) \geq 2$. Since G is bicritical, $G - \{x, y\}$ has a 1-factor or $\text{def}(G - \{x, y\}) = 0$, which is a contradiction.

(c) $\Rightarrow$ (b). Suppose that G is elementary with only singleton classes in $\mathcal{P}(G)$. Since G has 1-factors, by Tutte's 1-Factor Theorem, $o(G - S) \leq |S|$ for all $S \subseteq V(G)$. For any vertex set S with $|S| \geq 2$, as S is not a barrier, then $o(G - S) < |S|$, or $o(G - S) \leq |S| - 2$ by parity.

(b) $\Rightarrow$ (a). Suppose, to the contrary, that G is not bicritical, i.e., there exist two distinct vertices $x, y \in V(G)$ so that $G' = G - \{x, y\}$ has no 1-factors. Thus, there exists a vertex set $S' \subseteq V(G')$ such that $o(G' - S') \geq |S'| + 1$. Let $S = S' \cup \{x, y\}$. Then $o(G - S) = o(G' - S') \geq |S'| + 1 = |S| - 1$, a contradiction to (b) as $|S| = |S'| + 2 \geq 2$. $\square$

Now we describe a procedure to decompose an elementary graph into a collection of smaller bicritical graphs and elementary bipartite graphs. Hence these two families of graphs can be used as “Lego blocks” to construct all elementary graphs. This decomposition is called *bicritical decomposition*, which is a recursive process by contracting the components of the graphs along a maximal barrier.

Given an elementary graph G , choose a maximal barrier S with $|S| \geq 2$. For each component H of $G - S$, we conduct the following operations.

Contracting Process:

(C1) Contract each component of $G - S$ to a single vertex and then delete all edges spanned by S . Denote the resulting bipartite graph by G'_S ;

(C2) Contract the set $V(G) - V(H)$ to a single vertex u_H . Then the new graph with vertex set $V(H) \cup \{u_H\}$ is denoted by H' .

Clearly, from Theorem 5.1.7, we can repeat the process (C2) as long as H' (H' is elementary, see Theorem 5.1.8) is not bicritical (i.e., there exists a maximal barrier S with $|S| \geq 2$). Eventually, the graph G is broken into a list of elementary bipartite graphs and bicritical graphs, which is a bicritical decomposition of G . On the other hand, to reverse the above process or to generate all elementary graphs, we introduce the following method.

Construction Method: Let $G_0 = (U, W)$ be an elementary bipartite graph with more than two vertices. With each vertex $w \in W$ associate a (previously constructed) elementary graph $G(w)$ which has a property that $\mathcal{P}(G(w))$ contains a one-element equivalence class $\{v_w\}$ such that $d_{G(w)}(v_w) = d_{G_0}(w)$. Now splice $G(w)$ to G_0 at v_w and w as follows. Let

$A(w)$ and $B(w)$ denote the set of vertices joined to w in G_0 and to v_w in $G(w)$, respectively. Now delete v_w , w and construct any bipartite graph on $(A(w), B(w))$ satisfying that every vertex of $A(w) \cup B(w)$ is covered by as many new edges of this newly constructed bipartite graph as the number of deleted edges which covered it. Repeat this construction for each $w \in W$. Finally, add edges joining pairs of vertices in U arbitrarily.

The validity of contracting process and construction process are assured by Theorems 5.1.8 and 5.1.9. The proofs are omitted.

Theorem 5.1.8. *Let G be an elementary graph and $S \in \mathcal{P}(G)$ with $|S| \geq 2$. Let H be any component of $G - S$. Then, after conducting (C1) and (C2), we have*

- (a) *the graph G'_S is elementary;*
- (b) *the graph H' is elementary;*
- (c) $\mathcal{P}(H') = \{\{u_H\}\} \cup \{T \cap V(H) \mid T \in \mathcal{P}(G)\}.$

Theorem 5.1.9. *A graph is elementary if and only if it can be built up from elementary bipartite graphs and bicritical graphs by iterating the Construction Method.*

Is the collection of bicritical graphs and bipartite elementary graphs obtained by the bicritical decomposition unique or independent of the choice of S in the procedure? The answer is “No”. However, if the decomposition is pushed one step further, i.e., decompose the elementary graph into bricks (i.e., bicritical and 3-connected) and braces, then the number of bricks is invariant and furthermore the collection of bricks is uniquely determined, which is investigated in Section 5.3.

The 1-extendable graphs can be considered as a covering problem, i.e., there exists a set of 1-factors in a graph G to cover all edges of G . Then naturally we would like to have as fewer of 1-factors to cover the graph as possible. Based on this consideration, Berge posed the following conjecture.

Conjecture 5.1.1 (Berge, see [523]). *Let G be a 2-connected cubic graph. Then the edge set $E(G)$ can be covered by at most five 1-factors.*

The number of 1-factors used in this conjecture can not be reduced. For instance, the edge set of the Petersen graph can not be covered by fewer than five 1-factors, and any four perfect matchings can cover at most fourteen of its fifteen edges, but not all of them. The Conjecture 5.1.1 can be strengthened to the following famous conjecture from Fulkerson.

Conjecture 5.1.2 (Fulkerson (1971)). *Every edge of a 2-edge-connected cubic graph can be covered exactly twice by six 1-factors. That is, in a 2-edge-connected cubic graph, there exist six 1-factors to form a double covering.*

As an example, consider the cube $Q_3 = C_4 \square K_2$, which is a cubic bipartite graph and it can be decomposed into three disjoint 1-factors. Thus two copies of these three 1-factors form a double cover. It is an interesting exercise to find a double 1-factor covering for the Petersen graph. Actually, Fulkerson's Conjecture holds for all the generalized Petersen graphs, since Castagna and Prins (1972) showed that all of the generalized Petersen graphs are of class one except the Petersen graph itself.

Fulkerson's Conjecture is a long standing open problem, which has many connections with other graph theory problems and combinatorial optimization problems, e.g., the matching polytope problem. Moreover, if Fulkerson's Conjecture is true, then deleting one of the perfect matchings from the double cover would result in a covering of the graph by five 1-factors. So Berge's Conjecture is a weaker version of this conjecture. Fulkerson's Conjecture also implies another famous conjecture — Cycle Double Cover Conjecture. In fact, Fulkerson's Conjecture can be reformulated (Jaeger, 1988) in the following version.

Conjecture 5.1.3 (Fulkerson's Conjecture). *Every 2-edge-connected graph has a 6-cycle cover such that each edge is in exactly four cycles.*

Recently, Patel [472] initiated a different approach to the matching covering, by considering the maximum proportion of edges in a graph G that can be covered by k 1-factors, for $k \leq 6$. For a cubic 2-edge-connected graph G , let $\mathcal{M}$ be the set of distinct 1-factors in G . For a fixed positive integer k , define

$$m_k(G) = \max_{M_1, \dots, M_k \in \mathcal{M}} \frac{|\cup_{i=1}^k M_i|}{|E(G)|},$$

and

$$m_k = \inf_G m_k(G),$$

where the infimum is taken over all cubic 2-edge-connected graphs. Thus, for every cubic 2-edge-connected graph G and every positive integer k , G has a set of k 1-factors covering at least a proportion m_k of its edges. For example, let $\hat{P}$ be the Petersen graph, then it is not hard to verify $m_k(\hat{P}) = 1 - \binom{4}{k} / \binom{6}{k}$, $1 \leq k \leq 5$. He further made the following conjecture.

Conjecture 5.1.4 (Patel [472]). $m_k = m_k(\hat{P})$ for $1 \leq k \leq 5$.

For this conjecture, the case $k = 1$ is trivially true; the case $k = 2$ was proved by Kaiser, Král and Norine (2005); the case $k = 5$ is actually the Conjecture 5.1.1 by Berge. In the same paper, he also showed that Conjecture 5.1.2 implies Conjecture 5.1.4 by using a probabilistic argument: assume that a 2-connected cubic graph G has a double cover by six perfect matchings. If k of these six 1-factors are selected uniformly at random ($1 \leq k \leq 5$), then any given edge of G belongs to at least one of the chosen matchings with probability $1 - \binom{4}{k} / \binom{6}{k}$.

5.2 Ear decomposition

Near the end of the previous section, we introduced a structural decomposition for the family of elementary graphs. In this section, we study decomposition theory for other families of graphs. In particular, we concentrate on a special type of decomposition — ear decomposition. This decomposition is a recursive process and it gives not only a structure of the graphs and it can often be converted into an efficient algorithm conveniently.

Let G be a connected graph and G_0 a subgraph of G . An *ear* of G relative to G_0 is a path $P = v_1v_2 \dots v_k$ whose internal vertices (if any) have degree two in G , i.e., P meets G_0 exactly in its ends. In particular, the edge of P of length one is not an edge of G_0 . An ear is *odd* (or *even*) if the length of P is odd (or even). If P is an ear of G , then we denote by $G - P$ the graph obtained from G by deleting the edges and internal vertices of P . An ear P is *open* (or *closed*) if the end-vertices of P are distinct (or the same).

Let G be a graph and G_0 a subgraph of G . We express the ear decomposition of G starting with G_0 by a sequence of subgraphs of G : $G_0 \subset G_1 \subset G_2 \dots \subset G_m = G$, where $G_i = G_{i-1} + P_i$. That is, we proceed inductively to build a sequence of graphs as follow: if G_{r-1} has already been constructed, then we add an r th ear P_r to G_{r-1} , by joining two vertices in G_{r-1} with a path with no other vertex in common with G_{r-1} . From now on, in this section, we also use the notion $G_r = G_0 + P_1 + P_2 + \dots + P_r$ and $G_{r+1} = G_r + P_{r+1}$.

Let us start with an ear decomposition of 2-connected graph, which was described by Whitney (1932).

Theorem 5.2.1 (Whitney (1932)). *A graph is 2-connected if and only if it can be constructed from a cycle by successively adding an open ear to graphs previously constructed (see Fig. 5.2).*

Proof. Clearly, every graph constructed as described has no cutvertex, i.e., 2-connected.

Conversely, let G be a 2-connected graph. Then G contains a cycle, and hence has a (inclusionwise) *maximal* subgraph H constructible as described in the theorem. Since any edge $xy \in E(G) - E(H)$ with $x, y \in V(H)$ would define an ear, H is an induced subgraph of G . Thus, if $H \neq G$, then by the connectedness of G there is an edge $vw \in E(G)$ with $v \notin V(H)$ and $w \in V(H)$. As G is 2-connected, $G - \{w\}$ contains a shortest path P connecting v to H and then wvP is an open ear. Thus $H \cup \{wvP\}$ is a constructible subgraph of G larger than H , a contradiction to the maximality of H . $\square$

Note that the ears used in Theorem 5.2.1 can be even or odd. Similarly, we can describe all 2-edge-connected graphs by ear decompositions. This time, the ears used can be open or closed, odd or even. The proof is quite similar to that of Theorem 5.2.1.

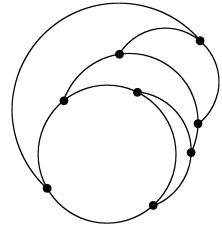


Fig. 5.2 The construction of 2-connected graph.

Theorem 5.2.2 (Whitney (1932)). *A graph is 2-edge-connected if and only if it can be constructed from a vertex by successively adding an ear to graphs previously constructed.*

Clearly, the number of ears in an ear decomposition of a 2-connected graph is independent of the decomposition and equals $|E| - |V| + 1$. Moreover, the similar ear decomposition has been obtained by Donald *et al.* [162] for strongly connected directed graphs as well (they used the term “handle bases” for the ears).

Theorem 5.2.3 (Donald *et al.* [162]). *A digraph D is strongly connected if and only if D has a directed ear decomposition $G_0 \subset G_1 \subset G_2 \cdots \subset G_m = D$, where G_0 is a direct cycle and $G_i = G_{i-1} + P_i$, and*

- (a) *each P_i is either open or closed directed ear;*
- (b) *P_i and P_j are arc disjoint, $i \neq j$.*

Furthermore, for every vertex v , every direct cycle C containing v can be used as the starting cycle G_0 .

Next, we consider the ear decomposition of factor-critical graphs. Lovász [400] observed that a factor-critical graph can be constructed from K_1 by iteration of the process of connecting two not necessarily distinct vertices of a previously constructed factor-critical graph by an odd ear.

Theorem 5.2.4 (Lovász [404]). *A graph G is factor-critical if and only if G has an ear decomposition $G = G_0 + P_1 + P_2 + \cdots + P_m$, where $G_0 = K_1$, satisfying*

- (a) *each P_r is either open or closed odd ear;*
- (b) *G_r is a factor-critical graph, $1 \leq r \leq m$.*

Furthermore, for any edge e of G , there is an odd ear decomposition of G such that the first ear uses e .

Proof. If G is factor-critical, by Theorem 6.1.12, G is connected, $G = K_1$ or $\delta(G) \geq 2$. We use induction on $|V(G)|$. Assume that G_r is already constructed and G_r is factor-critical. Without loss of generality, we assume that G_r is an induced subgraph. Otherwise, each edge e of $E(G) - E(G_r)$ with both ends

in G_r is an odd ear and $G_r \cup \{e\}$ is again factor-critical. If $G_r \neq G$, then $H = G - V(G_r)$ is a subgraph of even order and there exists a vertex x of H adjacent to a vertex u of G_r . Since G is factor-critical, there exist 1-factors M_u and M_x in $G - u$ and $G - x$, respectively. Then the symmetric difference $M_u \triangle M_x$ contains a path P of even length connecting u and x . Traversing along P from x to u , let the first vertex encountering G_r be v . Then the subpath P' of P from x to v has even length as $H - V(P')$ must be a graph of even order. Hence, uP' is an odd ear of G_r with end-vertices u and v . Note that this ear may be open or closed since u and v may be distinct or the same.

Next we show that $G_{r+1} = G_r + P_{r+1}$ is factor-critical. If the length of P_{r+1} is one, it is immediate. So assume that the length of P_{r+1} is at least three. Let $P_{r+1} = ux_1 \dots x_{2p}v$, where $u, v \in V(G_r)$. Since G_r is factor-critical, we need only to prove that $G_{r+1} - x_i$, $1 \leq i \leq 2p$, has a 1-factor. If i is odd, let M_1 be a 1-factor of $G_r - v$, then $M_1 \cup \{x_1x_2, \dots, x_{i-2}x_{i-1}, x_{i+1}x_{i+2}, \dots, x_{2p}v\}$ is a 1-factor of $G_{r+1} - x_i$; similarly, if i is even, let M_2 be a 1-factor of $G_r - u$, then $M_2 \cup \{ux_1, \dots, x_{i-2}x_{i-1}, x_{i+1}x_{i+2}, \dots, x_{2p-1}x_{2p}\}$ is a 1-factor of $G_{r+1} - x_i$. Thus G_{r+1} is factor-critical.

The sufficiency is easy to see, since all the G_r are factor-critical, in particular, $G = G_m$ is factor-critical. $\square$

In the above ear decomposition, we may need closed ears (i.e., odd cycles) in the process since there exist 1-connected factor-critical graphs. However, we know that a graph is factor-critical if and only if it is connected and every block is factor-critical. So a natural question is: if a factor-critical graph has no cutvertices, can we avoid to use closed ears? The next theorem provides a positive answer to this question.

Theorem 5.2.5. *A 2-connected graph G is factor-critical if and only if G has an ear decomposition $G = P_0 + P_1 + P_2 + \dots + P_m$, where P_0 is an odd cycle and each P_r ($1 \leq r \leq m$) is an open odd ear such that G_r is factor-critical.*

There are more research on decomposition of factor-critical graphs, such as the end decomposition in Section 5.3.

Since the most ears we dealt in this section are open odd ears, so from this point on, we simply refer open odd ears as ears, unless otherwise stated.

Below we give an ear decomposition theorem for elementary graphs.

Theorem 5.2.6 (Lovász [400]). *A graph G is elementary if and only if G has an ear decomposition $G = K_2 + P_1 + P_2 + \dots + P_m$, where P_r ($1 \leq r \leq m$) is an odd ear joining any two vertices x and y if $G_{r-1} - \{x, y\}$ has a 1-factor.*

Now we turn our attention to 1-extendable graphs to develop ear decomposition theory. We start the investigation with a simple family — bipartite graphs. Recall that in bipartite graphs, G is elementary if and only if it is 1-extendable.

Theorem 5.2.7 (Bipartite Ear Decomposition Theorem, Heteyi (1964)). *A bipartite graph G is 1-extendable if and only if G has an bipartite ear decomposition, i.e., there exists a sequence*

$$K_2 = G_0 \subset G_1 \subset \cdots \subset G_m = G$$

of bipartite subgraphs of G , for $i = 1, \dots, m$, $G_i = G_{i-1} + P_i$, where P_i is an ear of G_i joining two vertices in different color classes of G_{i-1} .

Proof. To see the sufficiency, we show that graphs of the described form are indeed 1-extendable. Clearly they are bipartite and connected. We proceed by induction on the number of ears.

Suppose $G = G_m \supseteq \dots G_1 \supseteq K_2 = G_0$ is as described in the theorem. If $i = 1$, G_i is an even cycle and hence 1-extendable. Assume that G_{i-1} is 1-extendable, we show that $G_i = G_{i-1} + P_i$ is also 1-extendable. Let the end-vertices of P_i be a and b , and $P_i = au_1u_2 \dots u_{2k}b$.

Choose any edge $e' \in E(G_{i-1})$. Since G_{i-1} is 1-extendable by the induction hypothesis, there is a 1-factor F of G_{i-1} containing e' and clearly F can be extended to a 1-factor of G_i containing e' . Hence all the edges of G_{i-1} and edges $\{u_1u_2, u_3u_4, \dots, u_{2k-1}u_{2k}\}$ are allowed in G_i .

Since G_{i-1} is 1-extendable, by Theorem 5.1.1 (d), $G_{i-1} - \{a, b\}$ has a 1-factor F' and F' can be extended to a 1-factor $F' \cup \{au_1, u_2u_3, \dots, u_{2k}b\}$ of G_i . Thus all edges of G_i lie in 1-factors and so G_i is 1-extendable.

Conversely, suppose that G is a 1-extendable bipartite graph. Choose any edge $e = xy$ and a 1-factor F_e of G containing e . Let $e' = yz$ be an edge adjacent to e and $F_{e'}$ be a 1-factor of G containing e' . The component of $F_e \cup F_{e'}$ containing e and e' is an even cycle $C = e + P_1$ and we have the first ear P_1 .

If $G = e + P_1$, we are done. So suppose there is an edge $f = ab$ in $G - E(P_1 \cup \{e\})$ with at least one end-vertex, say a , on $P_1 \cup \{e\}$. Let F_f be a 1-factor of G containing f . Define P to be the alternating path in $F_e \cup F_f$ obtained by starting at a , traversing $F_e \cup F_f$ through f to b and ending upon the first return to $P_1 \cup \{e\}$. The (necessarily odd) path so traversed is our second ear P_2 . Obviously, P_2 joined two vertices of $P_1 \cup \{e\}$ belonging to different color classes, as G is bipartite and P_2 is of odd length. Clearly we may continue to find new ears until every edge of G lies in some ear. $\square$

In fact, in the proof of the sufficiency, we showed a stronger statement that each G_i is 1-extendable, for $1 \leq i \leq m$.

From the above theorems, we see that factor-critical graphs or 1-extendable bipartite graphs can be constructed in a simple way by adding one ear at a time to a previously constructed graph. For the non-bipartite 1-extendable graphs, we can construct it in a similar way, but unfortunately we can not always insure that G_i is 1-extendable by adding a *single* ear at each step. For example, consider $G = K_4$ and let G_0 be any edge. After adding an ear of

length 3, we need to add both of the remaining two edges in next step to get a 1-extendable graph. So we have to modify the definition of ear decomposition to address this new issue.

Consider a generalized ear decomposition of G starting with G_0 . For this decomposition, we use a representation in the form $G_0 \subset G_1 \cdots \subset G_m = G$, where $G_i = G_{i-1} + R_i$, and R_i is a *set of ears* instead of a single ear. So the ear decomposition studied earlier is a special case of this more general concept. As usual, we denote $G_i = G_0 + R_1 + \cdots + R_i$. Our goal is to insure that each G_i is 1-extendable. A necessary condition to maintain the continuation of such ear decomposition is that $G - V(G_i)$ still have a 1-factor. So we introduce a term — niceness. A subgraph G' in G is called *nice* if $G - V(G')$ has a 1-factor.

A *graded ear decomposition* of G is a sequence of subgraphs $G_0 \subset G_1 \subset \cdots \subset G_m = G$ such that for each i , G_i is a nice 1-extendable subgraph of G , where G_i is obtained from G_{i-1} by attaching a set of ears.

Next theorem guaranties the existence of a graded ear decomposition for all 1-extendable graphs.

Theorem 5.2.8 (Lovász and Plummer [412]). *Every 1-extendable graph G has a graded ear decomposition starting with any nice 1-extendable subgraph G_0 .*

Proof. Let G and G_0 be as given in the theorem and suppose $G_i \neq G$ has been constructed. Without loss of generality, assume that G_i does not span G , otherwise we add all remaining edges to G_i to obtain G_{i+1} . Since G is connected, we can choose an edge e joining $V(G_i)$ to $V(G) - V(G_i)$. Let M be any 1-factor of $G - V(G_i)$ and let F be a 1-factor of G containing e . So $M \triangle F$ consists of disjoint alternating paths and alternating cycles. We add all the alternating paths in $M \triangle F$ to G_i to obtain G_{i+1} , denote $G_i = G_{i-1} + R_i$, where R_i is the set of such alternating paths. Clearly, such path must exist, since there is at least one such path in $M \triangle F$, namely that containing the edge e . Continue in this manner, we eventually reach a first spanning subgraph of G , say G^* . Then add all the remaining edges to obtain G .

The remaining of the proof is to show that G_{i+1} is a nice 1-extendable subgraph of G . To see that G_{i+1} is nice, it is easy to see that $M \cap E(G - V(G_{i+1}))$ is a 1-factor of $G - V(G_{i+1})$.

Next to show that G_{i+1} is 1-extendable. Let e' be any edge of G_i and F' be a 1-factor of G_i containing e' . Then $F' \cup (M \cap R_i)$ is a 1-factor of G_i containing e' . Next, let e' be an edge of $F \cap R_i$. From the construction of G_{i+1} above, e' belongs to the 1-factor F and so F restricting on G_{i+1} is a 1-factor of G_{i+1} containing e' . That is, G_{i+1} is 1-extendable. $\square$

We commented earlier that to maintain 1-extendability in each step of ear decomposition, we may have to add more than one ears to the previous

constructed subgraph in the sequence. However, how many ears are required at each step? Naturally, the longer the sequence of a decomposition is, the fewer ears are required at each step. More precisely, if we consider the *longest* (i.e., largest m among all graded ear decompositions) or *non-refinable* graded ear decompositions (i.e., if $G_{i+1} = G_i + R_1 + R_2 + \cdots + R_k$, then the proper subgraph $G_i + R_{i_1} + \cdots + R_{i_r}$ is not 1-extendable for any r , $0 < r < k$), then we never need more than two ears from G_i to G_{i+1} to maintain 1-extendability of G_{i+1} . This statement is the so-called “Two Ears Theorem” first proved by Lovász [407] and different proofs are provided by Little and Rendl [350], Szigeti [541], and de Carvalho, Lucchesi and Murty [116].

The original proof of Two Ears Theorem is quite involved and relies on a lot of preparation. Here we use the simple and short proof from Szigeti (1998), which relies only on Gallai-Edmonds Structure Theorem. For the convenience of reading, we collect the results used in the proof below as lemmas and they all can be deduced from Gallai-Edmonds Structure Theorem easily.

A barrier X of a graph G is called *strong* if each odd component of $G - X$ is factor-critical.

Lemma 5.2.1 (Claim 2 of the proof of Theorem 1.5.7). *Let G be an elementary graph and X a strong barrier. Then each edge leaving X belongs to some 1-factor of G .*

Lemma 5.2.2. *Let G be an elementary graph and let e_1, e_2, e_3 be edges not in $E(G)$ but having both end-vertices in $V(G)$. If $G \cup \{e_1, e_2, e_3\}$ has a 1-factor M containing $\{e_1, e_2, e_3\}$, and for each e_i ($1 \leq i \leq 3$), e_i is forbidden in $G \cup \{e_i\}$, then for each e_i , there exists an e_j ($1 \leq j \leq 3, i \neq j$), such that $G \cup \{e_i, e_j\}$ has a 1-factor containing e_i and e_j .*

Proof. Let $e_i = x_i y_i$ ($1 \leq i \leq 3$). Assume that $G \cup \{e_1, e_2\}$ has no 1-factors containing e_1 and e_2 , we shall prove $G \cup \{e_1, e_3\}$ has a 1-factor containing e_1 and e_3 . The conclusion follows from the following facts.

(1) *There exists a strong barrier S in $G \cup \{e_1, e_2\}$ containing x_1 and y_1 .* Clearly, $G \cup \{e_1, e_2\}$ has a 1-factor, and e_1 is forbidden in $G \cup \{e_1, e_2\}$ since $G \cup \{e_1, e_2\}$ has no 1-factors containing $\{e_1, e_2\}$, and e_1 is forbidden in $G \cup \{e_1\}$. By Theorem 5.1.4 (d), there exists a barrier of $G \cup \{e_1, e_2\}$ containing x_1 and y_1 . Let S be a maximal such barrier. Then, by Theorem 5.1.4 (e), each component C_i of $G - S$ ($1 \leq i \leq |S|$) is factor-critical, i.e., S is a strong barrier.

(2) *Edge e_2 is in one of the factor-critical components (say, C_1) of $G \cup \{e_1, e_2\} - S$.* Indeed, by Lemma 5.2.1, e_2 does not enter S . Moreover, x_2 and y_2 can not be contained in S ; otherwise, $S - \{x_1, y_1, x_2, y_2\}$ violates the Tutte’s condition in $(G \cup e_3) - \{x_1, y_1, x_2, y_2\}$, contradicting the assumption that $G \cup \{e_1, e_2, e_3\}$ has the 1-factor M containing e_1, e_2 and e_3 .

(3) *x_3 and y_3 are in different factor-critical components of $G \cup \{e_1, e_2\} - S$.* This follows from the fact that $(G \cup \{e_1, e_2\} - x_1 - y_1) \cup e_3$ contains the 1-factor $M - e_1$. It also follows that

(4) for each C_i ($1 \leq i \leq |S|$) exactly one edge f_i of M leaves C_i in $G \cup \{e_1, e_2\}$.

Now, suppose that f_1 enters S . Since S is a strong barrier in $G \cup \{e_2\}$, by Lemma 5.2.1, f_1 belongs to a 1-factor M_1 of $G \cup \{e_2\}$. Then $(M \cap E(C_1)) \cup (M_1 - (M_1 \cap E(C_1)))$ is a 1-factor of $G \cup \{e_2\}$ containing e_2 . This contradiction shows that

(5) f_1 does not enter S . Moreover, in $G - \{e_1, e_2, e_3\}$ e_3 leaves the factor-critical component of $G - \{e_1, e_2\} - S$ that contains e_2 ; that is $f_1 = e_3$. Assume, without loss of generality, that x_3 is in C_1 . Then $H = C_1 - \{x_3\}$ has a 1-factor, for example $M \cap E(H)$.

(6) $H - \{e_2\}$ has a 1-factor M_2 . Otherwise, for a maximal barrier X of $H - \{e_2\}$, we have $o(H - \{e_2\} - X) \geq |X| + 2$. Then, $S' = S \cup X \cup \{x_3\}$ is a strong barrier in $G \cup \{e_3\}$, and e_3 enters S' ; thus by Lemma 5.2.1 $G \cup \{e_3\}$ contains a 1-factor containing e_3 , a contradiction.

(7) $M \cap E(G \cup \{e_1, e_2, e_3\} - H) \cup M_2$ is a 1-factor of $G \cup \{e_1, e_3\}$ containing e_1 and e_3 , as we desired. □

Now we are ready to prove the main theorem of the section.

Theorem 5.2.9 (Two Ears Theorem, Lovász [407]). *If G is a 1-extendable graph, then there exists a graded ear decomposition $K_2 = G_0 \subset G_1 \cdots \subset G_m = G$, where $G_i = G_{i-1} + R_i$, such that for each i , $0 \leq i < m$, G_i is 1-extendable and R_i has at most two ears.*

Proof. Let G be a 1-extendable graph and $e_1, e_2, \dots, e_k$ be edges not in $E(G)$ but having both end-vertices in $V(G)$.

Claim 1. Suppose that $\Phi(G \cup \{e_1, \dots, e_k\}) > \Phi(G)$. Then there exist i and j , $1 \leq i \leq j \leq k$ such that $\Phi(G \cup \{e_i, e_j\}) > \Phi(G)$.

Without loss of generality, we may assume that $\{e_1, \dots, e_k\}$ is a minimal set such that $\Phi(G \cup \{e_1, \dots, e_k\}) > \Phi(G)$. Then we show that $k \leq 2$. Assume that $k \geq 3$ and let $G^* = G \cup \{e_4, \dots, e_k\}$. Then, by the minimality, $\Phi(G^*) = \Phi(G)$ and $\Phi(G^* \cup \{e_i\}) = \Phi(G^*)$ ($i = 1, 2, 3$), but $\Phi(G^* \cup \{e_1, e_2, e_3\}) > \Phi(G) = \Phi(G^*)$. Thus Lemma 5.2.2 implies that for some $1 \leq i < j \leq 3$, $\Phi(G^* \cup \{e_i, e_j\}) > \Phi(G^*)$; that is, $\Phi(G \cup \{e_i, e_j, e_4, \dots, e_k\}) > \Phi(G)$, contradicting to the minimality. Hence $k \leq 2$.

Based on Claim 1, now we prove the theorem. Assume that, for some r , G_r has already been constructed and $G_{r+1} = G_r + P_1 + \dots + P_k$, where G_r and G_{r+1} are nice 1-extendable graphs, and P_i are ears. Define e_i to be the edge connecting the two end-vertices of P_i for $i = 1, \dots, k$. Clearly, for any subset $\{P_{i_1}, \dots, P_{i_m}\}$ of $\{P_1, \dots, P_k\}$, $G_r + P_{i_1} + \dots + P_{i_m}$ is 1-extendable if and only if $G_r \cup \{e_{i_1}, \dots, e_{i_m}\}$ is 1-extendable. Thus Claim 1 implies the theorem. □

Remark 5.2.1. (a) Based on Theorems 5.2.8 and 5.2.9, we see that every 1-extendable graph has a non-refinable graded ear decomposition starting with an arbitrary edge (or with an even cycle), in which each of grade G_{r+1} results from the previous G_r through the addition of *single* or *double* ears. In the case of a single-ear addition, the ear must join two different classes of $\mathcal{P}(G_r)$, whereas when a double-ear addition is employed, one ear joins two vertices of one class and the other joins two vertices of a different class of $\mathcal{P}(G_r)$. On the other hand, an arbitrary double-ear addition to a 1-extendable graph may not result in a new 1-extendable graph.

(b) When G is a 1-extendable *bipartite* graph, from Theorem 5.2.7, we start with a single edge and at each step only add single-ear to maintain 1-extendability of G_r . However, for non-bipartite 1-extendable graph, we still start with a single edge but at a point we have to add a double-ear. The first double-ear addition results in a non-bipartite graph. How soon do we have to add a double-ear? It was shown that a double-ear addition can happen pretty soon (see Theorem 5.4.10, [412]). Assume G is a non-bipartite 1-extendable graph, then G has a non-refinable graded ear decomposition $K_2 = G_0 \subset G_1 \subset G_2 \subset G_3 \cdots \subset G_m = G$, where G_1 is a nice cycle, $G_2 = G_1 + R_2$ and $G_3 = G_2 + R_3$, either R_2 or R_3 is a double ear.

(c) For a non-bipartite 1-extendable graph G , G has an ear decomposition in which $G_0 = K_2$, either G_2 is an odd subdivision of K_4 or G_3 is an odd subdivision of $\overline{C}_6$, which is the complement of the 6-cycle, i.e., $\overline{C}_6 = K_3 \square K_2$ (see Lovász [407] for details).

Two Ears Theorem insures the existence of graded ear decompositions with at most two ears only at each step. Recently, de Carvalho, Lucchesi and Murty [116] generalized Theorem 5.2.9 and find a lower bound on the number of such ear decompositions.

Theorem 5.2.10 (de Carvalho, Lucchesi and Murty [116]). *Every 1-extendable graph G with maximum degree Δ has at least $\Delta!$ such graded ear decompositions.*

For bricks, we have the following nice result, which provides an answer to a question raised by Naddef and Pulleyblank [440].

Theorem 5.2.11 (de Carvalho, Lucchesi and Murty [117]). *Every brick that is not a Petersen brick has an ear decomposition with exactly one double-ear.*

We mentioned the possible locations of double-ears in Remark 5.2.1 (a). Recently de Carvalho and Cheriyan [113] give a further clarification of the locations of double-ears, as seen from the next two theorems.

Theorem 5.2.12 (de Carvalho and Cheriyan [113]). *Let G be a 1-extendable graph, and $e, f \notin E(G)$, but having both end-vertices in $V(G)$. Assume that both $G \cup \{e\}$ and $G \cup \{f\}$ are not 1-extendable, then $\{e, f\}$ is*

a double-ear addition relative to G if and only if the ends of f lie in distinct classes of the canonical partition $\mathcal{P}(G \cup \{e\})$.

Proof. Suppose that $\{e, f\}$ is a double-ear addition relative to G . Then, $G \cup \{e, f\}$ is a 1-extendable graph, and so the edge f is allowed in $G \cup \{e, f\}$. It follows from Theorem 5.1.6 (c) that the ends of f are in distinct classes of $\mathcal{P}(G \cup \{e\})$.

Conversely, if the ends of f are in distinct classes of $\mathcal{P}(G \cup \{e\})$, then f is allowed in $G \cup \{e, f\}$ by Theorem 5.1.6 (c) again. But note that f is forbidden in $G \cup \{f\}$. Similarly, e is forbidden in $G \cup \{e\}$. Thus, every 1-factor in $G \cup \{e, f\}$ containing one of e and f must contain the other, and so $G \cup \{e, f\}$ is 1-extendable, i.e., $\{e, f\}$ is a double-ear relative to G . $\square$

Theorem 5.2.13 (de Carvalho and Cheriyan [113]). *Let G be a 1-extendable graph, and $e_i \notin E(G)$, but having both end-vertices in $V(G)$ for each $i = 1, \dots, k$. Suppose that $G \cup \{e_1, \dots, e_k\}$ is 1-extendable but $G \cup \{e_i\}$ is not ($1 \leq i \leq k$). If $S \in \mathcal{P}(G)$ and there exists precisely one edge in $\{e_1, \dots, e_k\}$, say $e_1 = xy$, such that $\{x, y\} \subseteq S$, then e_1 must be a member of a double-ear relative to G , that is, $\{e_1, e_j\}$ is a double-ear for some $j \in \{2, \dots, k\}$.*

The investigation of locations of double-ears enables de Carvalho and Cheriyan [113] to design a linear time algorithm to find a double-ear and thus find a best known algorithm with complexity $O(|V||E|)$ for constructing an ear decomposition for 1-extendable graphs. Their algorithm is based on two efficient $O(|V||E|)$ -time algorithms to solve two other fundamental problems in matching theory: (1) finding all the allowed edges of a graph; (2) finding the canonical partition of an elementary graph.

To avoid double-ear additions, Lou, Saito and Teng [392] take a different approach and modify the process of adding ears. They give a characterization of 1-extendable graphs in terms of a special alternating paths. This decomposition enables them to find an efficient algorithm to determine whether an input graph G is 1-extendable.

Theorem 5.2.14 (Lou, Saito and Teng [392]). *Let G be a graph with a 1-factor M_0 . Then G is 1-extendable if and only if for any pair of vertices x and y with an M_0 -alternating xy -path P_0 of length three which starts with an edge that belongs to M_0 (see Fig. 5.3), there exists an M_0 -alternating path P connecting x and y , of which the starting and the ending edges are not in M_0 .*

It is well-known that any two edges of a 2-connected graph lie on a common cycle. This statement holds for 1-extendable graphs as well since 1-extendable graphs are always 2-connected. However, Little (1974) proved a stronger result that any two edges of a 1-extendable graph lie on a *nice* cycle. A short

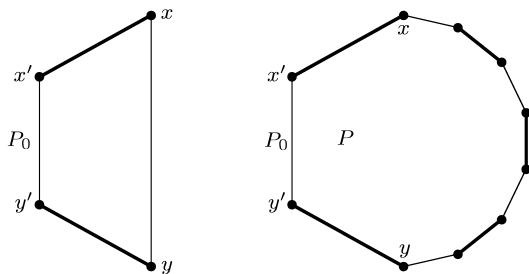


Fig. 5.3 Two M -alternating paths between x and y .

proof given here is due to Lovász and Plummer [412] by applying the technique of graded ear decompositions.

Theorem 5.2.15 (Little [348]). *If G is a 1-extendable graph and e_1 and e_2 are any two edges of G , then e_1 and e_2 lie on a nice cycle.*

Proof. Use induction on $|E(G)|$.

First suppose that both end-vertices of e_2 are of degree two. Then replace the path of length 3, having e_2 as its middle edge, by a single edge e_3 . The new graph G^* resulting from this operation is 1-extendable and has $|E(G^*)| < |E(G)|$. So applying the induction hypothesis to G^* , there is a nice cycle in G^* containing e_1 and e_3 and this cycle clearly extends to a nice cycle in G containing e_1 and e_2 .

So assume e_2 has an end-vertex of degree at least three. Let F_2 be a 1-factor in G containing e_2 . We claim that there is a cycle C containing e_1 and alternating with respect to F_2 . If $e_1 \in F_2$, let e_3 be any edge adjacent to e_1 and F_3 a 1-factor containing e_3 . Then let C be the F_2 - F_3 alternating cycle containing e_1 . If $e_1 \notin F_2$, let F_1 be any 1-factor containing e_1 and let C be the F_1 - F_2 alternating cycle containing e_1 . This proves the claim.

From Theorem 5.2.8, we have a graded ear decomposition of G starting with e_1 , $\{e_1\} = G_0 \subset G_1 = C \subset G_2 \subset \cdots \subset G_m = G$. From the proof of Theorem 5.2.8, we see that if F is any 1-factor of $G - V(G_r)$, then all ears after G_r can be chosen to be alternating with respect to F . In particular, consider $G_1 = C$ and F_2 , then all ears added after G_1 are alternating with respect to F_2 . Let G_r be the first graph in the sequence which includes e_2 . Then $G_r \neq G$, as e_2 (which is in F_2) has at least one end-vertex of degree at least 3 in G . Thus $|E(G_r)| < |E(G)|$ and so, by the induction hypothesis on G_r , there is an alternating cycle in G_r (and hence in G) containing e_1 and e_2 . $\square$

Ear decomposition is an important tool not only providing a recursive construction for a given family of graphs, also can be easily converted into an algorithm for finding this family of graphs. In this section, we have studied the ear decompositions for several families of graphs, e.g., factor-critical graphs,

elementary graphs and 1-extendable graphs. It is natural to find an ear decomposition for bicritical graphs or 2-extendable graphs. However, bicritical graphs are 3-edge-connected and 2-extendable graphs are 3-connected, so we can not add ears to maintain the connectivity at each step for those classes of graphs. Therefore new ideas are required to develop decomposition processes for these classes of graphs.

In Section 5.4, we return to ear decomposition and present the recent progresses on optimal ear decompositions (i.e., use the least possible number of double ears).

5.3 Minimal graphs and more decompositions

Brick decomposition is a pioneer work in the approach to determine the number of 1-factors in graphs. It was used to study the dimensions of matching polytopes and matching lattices. More recently, there are major advances in the study of brick decomposition, which is introduced in Section 5.4. In this section, we first discuss minimal factor-critical and bicritical graphs and its properties. At the second half of the section, we present End Decomposition introduced by Frank (1993) and Brick Decomposition from Lovász (1972).

A factor-critical (resp. bicritical) graph G is *minimal* if $G - e$ is not factor-critical (resp. bicritical) for any $e \in E(G)$.

Due to the importance of factor-critical graphs and bicritical graphs in matching theory, more properties of these families of graphs are desired. One typical approach is to investigate the minimal family since it tends to possess more properties. We list some known results in this direction, but far less than what we would like to know. The research in minimal such graphs remains as a virgin land to explore.

Next, we demonstrate a few known classes of minimal factor-critical graphs and minimal bicritical graphs. For example, any triangle block graph (that is, a connected graph with each block being a triangle) is a minimal factor-critical graph. The even wheels, $K_1 \vee C_{2k-1}$, and prisms, $K_3 \square K_2$, are minimal bicritical graphs.

To construct more minimal bicritical graphs, we turn to the well-known Halin graphs. A graph H is a *Halin graph* if it can be constructed as follows: let T be any tree in which each non-leaf vertex has minimum degree 3 (see Fig. 5.4). Embed T in the plane and construct a cycle C through all the leaves of T in such a way that $H = T \cup C$ remains plane.

Theorem 5.3.1 (Lovász and Plummer [409]). *Every Halin graph of even order is minimal bicritical.*

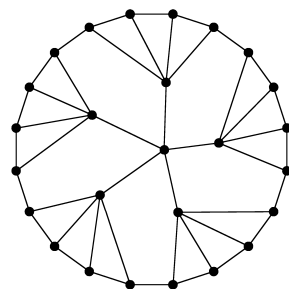


Fig. 5.4 A Halin graph.

Proof. Bondy and Lovász [87] showed that Halin graphs are 1-hamiltonian, i.e., for every Halin graph H , H has a hamiltonian cycle, and if any vertex is deleted, the resulting graph still has a hamiltonian cycle.

Thus, for any two vertices $u, v \in V(H)$, $H - \{v\}$ has a hamiltonian cycle and thus $H - \{u, v\}$ has a hamiltonian path of odd length, that is, $H - \{u, v\}$ has a 1-factor. Hence H is bicritical.

To see that H is minimal, let $e = uv$ be any edge of H . If e lies on the outer cycle C , then the vertices u and v are of degree 2 in $H - \{e\}$. Let x and y be the neighbors of u in $H - \{e\}$. Then u is an isolated vertex of $(H - \{e\}) - \{x, y\}$ or $H - \{e\}$ is not bicritical. If $e \in T$, then $T - \{e\}$ is disconnected and thus there exist two edges e_1 and e_2 on C such that $(H - \{e\}) - \{e_1, e_2\}$ is disconnected. Therefore, there are four 2-vertex-cuts formed by the end-vertices of e_1 and e_2 in $H - \{e\}$. Among the four cuts, there is at least one cut, say S , separating $H - \{e\}$ into two odd components. Then $(H - \{e\}) - S$ has no 1-factors or $H - \{e\}$ is not bicritical. Thus H is a minimal bicritical graph. $\square$

Theorem 5.3.2 (Lovász and Plummer [409]). *Let H be a Halin graph of even order n . Then $\Phi(H) \geq \frac{2}{3}(n - 1)$ unless H is the graph of Fig. 5.5. Moreover, this bound is best possible.*

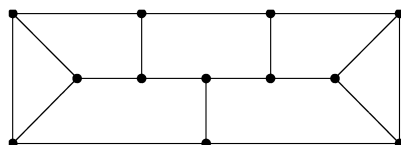


Fig. 5.5 An exceptional graph.

Next we present several results related to forbidden configurations in minimal graphs.

Theorem 5.3.3. *If G is a minimal factor-critical graph, then*

- (a) *every nice factor-critical subgraph of G is also minimal factor-critical;*
- (b) *if G contains a K_3 , then this K_3 must be a block of G ;*

(c) G does not contain a C_4 .

Theorem 5.3.4. *If H is a forbidden subgraph in all minimal bicritical graphs, it is also excluded in all minimal factor-critical graphs.*

Theorem 5.3.5 (Lovász and Plummer [410]). *Let G be a minimal bicritical graph. Then*

- (a) *if G is not a wheel itself, then it contains no wheel as a subgraph;*
- (b) *G does not contain $K_{3,3}$ as a subgraph.*

Next we study the extremal structure of minimal factor-critical graphs and minimal bicritical graphs. In particular, the maximum possible edges in such graphs.

Theorem 5.3.6 (see [412]). *If G is a minimal factor-critical graph of order n , then*

$$|E(G)| \leq \frac{3(n-1)}{2}.$$

Moreover, the equality holds if and only if G is a connected graph, and each block of which is a K_3 .

Theorem 5.3.7 (Lovász and Plummer [410]). *If G is a minimal bicritical graph of order $n \geq 6$, then*

$$|E(G)| \leq \frac{5(n-2)}{2}.$$

Moreover, the equality holds if and only if G is a join of $\frac{n-2}{2}K_2$ and $2K_1$.

The original proof of Theorem 5.3.7 heavily relies on ear decompositions of the graphs. It is interesting to give a direct proof of the above theorem without using ear decompositions.

For minimal bricks, Norine and Thomas (2006) find a sharp bound on $|E(G)|$.

Theorem 5.3.8 (Norine and Thomas [467]). *Let G be a minimal brick of order n . Then*

$$|E(G)| \leq \frac{5n-14}{2},$$

unless G is the prism or the wheels W_m , where $m = 3, 5, 7$.

Theorem 5.3.6 and Theorem 5.3.7 also suggest the following conjecture.

Conjecture 5.3.1. *If G is a minimal k -factor-critical graph of order n , then*

$$|E(G)| \leq \frac{(2k+1)(n-k)}{2}.$$

Bricks play important roles in the study of matchings, for instance, brick decomposition and links with the dimension of matching polytopes. Below we present a few properties of bricks obtained by Lovász [408], but more discussion takes place in the next section. Denote by $\nabla(S)$ the set of edges in G connecting S to $V(G) - S$. An edge cut $\nabla(S)$ is called *odd* if $|S|$ is odd.

Theorem 5.3.9 (Lovász [408]). *Let G be a brick and $C = \nabla(S)$ be an odd cut in G . Let G_S be a graph obtained from G by contracting S to a single vertex s . Then G_S is a brick if and only if $G - S$ is factor-critical and 2-connected.*

Proof. Suppose that G_S is a brick. If $G - S = G_S - \{s\}$ has a cutvertex x , then $\{x, s\}$ is a 2-cut set of G_S , a contradiction. For any vertex $u \in V(G) - S = V(G_S) - \{s\}$, then $(G - S) - \{u\} = (G_S - \{s\}) - \{u\}$ has a 1-factor, i.e., $G - S$ is factor-critical.

Conversely, assume that $G - S$ is a 2-connected factor-critical graph. Then G_S is 3-connected. For, a 2-cut in G_S containing s corresponds to a cutvertex in $G - S$, while one not containing s also separates G . Similarly, a non-trivial barrier in G_S containing s contains a vertex of $G - S$ whose removal does not leave a 1-factor, while one not containing s also remains a barrier in G . $\square$

Theorem 5.3.10 (Lovász [408]). *Let G be a brick different from K_4 and $\overline{C}_6$. Then G has an edge e such that $G - e$ is 1-extendable.*

Next we consider the vertices with the minimum degree in minimal graphs. From Theorems 6.1.11 and 6.1.12, the minimum degrees of 1-extendable graphs and bicritical graphs are at least 2 and 3, respectively. So, what happen to minimal graphs?

Theorem 5.3.11. *Every minimal elementary graph (i.e., minimal 1-extendable graph) contains at least two adjacent vertices of degree two.*

de Carvalho, Lucchesi and Murty [124] proved that every minimal brick has a vertex of degree three. Norine and Thomas strengthen this result to the following.

Theorem 5.3.12 (Norine and Thomas [467]). *Every minimal brick has at least three vertices of degree three.*

For minimal bricks, we expect a similar result as that of Theorem 5.3.11. We enclose a long-standing conjecture from Lovász.

Conjecture 5.3.2 (Lovász (1973)). *Every minimal brick has two adjacent vertices of degree three.*

Hence, although bricks play important roles in studying matchings in graphs and in determining the dimensions of perfect matching polytopes and matching lattices, but very little is known about the structures of bricks,

even the vertices with the minimum degree. However, there are some recent progresses in this front, we shall survey this topic in the next section.

Now we return to factor-critical graphs again. Given a non-factor-critical graph, how can we turn it into a factor-critical graph? How to measure how far a graph is from a factor-critical graph? Since every factor-critical graph is connected and 2-edge-connected, in order to become a factor-critical graph we need to conduct an operation to meet these necessary requirements. There are three possible operations to consider: contraction, subdivision and ear decomposition.

The contraction of an edge e of a graph is defined in the usual way. By the contraction of an edge set F of G , G/F , we mean it arising from G by contracting each edge of F . We always assume (without loss of generality) that this edge set is cycle-free, i.e., F is a forest. For a connected graph G , an edge set F is called *critical-making* if the contraction G/F results a factor-critical graph. So a minimal critical-making set can be used as a measurement for how close a graph is to factor-criticality. Since factor-critical graphs are 2-edge-connected, every cut edge of G is contained in any critical-making set. Thus we may assume that G is 2-edge-connected.

From Theorems 5.2.2 and 5.2.4, every 2-edge-connected graph has an ear decomposition and any factor-critical graph has an *open* ear decomposition. So the number of even ears can be used as a measurement for how close a graph is to factor-criticality too. Let $\varphi(G)$ denote the minimum number of even ears in ear decompositions of G . Clearly, a graph G is factor-critical if and only if $\varphi(G) = 0$. Moreover, for a bipartite graph G , $\varphi(G) = 1$ if and only if G is 1-extendable (equivalently, elementary). Heteyi (1964) proved that if G is 1-extendable, then $\varphi(G) = 1$. Frank [214] established a good characterization for general graphs having $\varphi(G) = 1$ and showed that $\varphi(G) = 1$ if and only if G has a 1-factor M and M is a maximum join (see the definition after Theorem 5.3.14).

The *subdivision* of an edge set F of a graph G means that we subdivide each edge e of F by a new vertex. The resulting graph is denoted by $G \succ F$.

Interesting enough, Szigeti (1996) showed that all the three approaches defined above are actually equivalent.

Theorem 5.3.13 (Szigeti [540]). *Let G be a 2-edge-connected graph and k be a positive integer. Then the following statements are equivalent:*

- (a) $\varphi(G) = k$;
- (b) $\min\{|F| \mid F \text{ is a critical-making set}\} = k$;
- (c) $\min\{|F| \mid G \succ F \text{ is a factor-critical graph}\} = k$.

Proof. (a) $\Leftrightarrow$ (c). Let $|F_0| = \min\{|F| \mid G \succ F \text{ is a factor-critical graph}\} = k$.

Since $G \succ F_0$ is factor-critical, it has an *open odd* ear decomposition by Theorem 5.2.4. The corresponding ear decomposition of G has exactly $|F_0| = k$ even ears, thus $\varphi(G) \leq k$.

Conversely, Let $G = P_0 + \cdots + P_m$ be an ear decomposition of G with $\varphi(G)$ even ears. Choose an edge from each even ear. Then the subdivision of these edges leaves a factor-critical graph, thus $\varphi(G) \geq k$ and we are done.

(b) $\Leftrightarrow$ (c). We prove the following statement by induction on k .

If $F \subseteq E(G)$ is a forest, $|F| = k$ and G/F is factor-critical, then $G \succ F$ is factor-critical.

When $k = 1$. Suppose, to the contrary, that for some edge f , G/f is factor-critical but $G \succ f$ is not. Since $G \succ f$ has an odd number of vertices and it is not factor-critical, by Gallai-Edmonds Structure Theorem, then $S = A(G \succ f) \neq \emptyset$ is a barrier, i.e., $o((G \succ f) - S) \geq |S| + 1$.

Let $f = uv$ and the insertion vertex on f be w . Note that G/f is obtained from $G \succ f$ by contracting edges uw and wv . Observing the changes of S and the odd components in the process of contracting uw and wv , the set S may increase by one more vertex or the number of odd components may decrease by one. It is easy but tedious to verify that regardless the locations of u, v, w , there exists a vertex set $X \neq \emptyset$ in G/f so that $(G/f) - X$ has at least $|X|$ odd components. Thus for a vertex $x \in X$, $(G/f) - x$ has no 1-factors, which contradicts the assumption that G/f is factor-critical.

Assume that the statement is true for $|F| \leq k-1$. Let $F \subseteq E(G)$ be a forest so that G/F is factor-critical and $|F| = k$. Let $f \in F$. Consider the graph $G' = G/f$ and let $F' = F - \{f\}$. Then G'/F' is factor-critical, F' is a forest and $|F'| = k-1$, thus by induction, $G' \succ F'$ is factor-critical. By the induction hypothesis for the graph $G \succ F'$ and the edge f , $(G \succ F') \succ f = G \succ F$ is factor-critical. So the statement is proved.

Conversely, assume that $G \succ F$ is factor-critical. Then by Theorem 5.2.4 it has an odd ear decomposition. Note that the insertion vertices must be internal vertices of the odd paths, thus the corresponding odd ear decomposition of G/F shows that G/F is factor-critical. $\square$

This theorem implies that $\varphi(G/F) = \varphi(G \succ F)$ for any forest F . In view of this fact, we shall use the subdivision of F rather than the contraction since it is easier to deal with subdivision than with contraction. Clearly, the subdivision of any edge in a graph decreases or increases $\varphi(G)$ by one. Thus $\varphi(G \succ F) \geq \varphi(G) - |F|$ for any edge set F of G .

Similar to bicritical decomposition at the end of Section 5.1, we now introduce another decomposition, the end decomposition, which is one whose building blocks are factor-critical graphs and elementary (i.e., 1-extendable) bipartite graphs. We show later that the length of any end decomposition is always φ .

For a graph H with a 1-factor, recall that a barrier S is called a *strong barrier* if (1) $H - S$ consists of exactly $|S|$ odd components; (2) no even component; (3) each odd component is factor-critical, and (4) the bipartite graph H_S associated with S is elementary (where H_S is obtained by deleting

the edges induced by S and contract each odd component). Each elementary graph has a strong barrier. In fact, each maximal barrier of an elementary graph is a strong barrier.

For a graph $G = (V, E)$, let $H = G[U]$ be a subgraph induced by $U \subseteq V$ so that H has a strong barrier, say S . We call H a *strong end* of G attached at S if S is a cut-set of G separating $U - S$ from $V - U$ or if $U = V$ (in which case G itself is a strong barrier). It is easy to see that a factor-critical graph has no strong ends. It can be shown that any non-factor-critical graph has a strong end. With this property, we can implement an procedure to break down a non-factor-critical graph into factor-critical blocks.

Given a graph $G = (V, E)$, let $H = G(U)$ be a strong end of G . An *end decomposition* of G is a sequence $(G_0, U_0), (G_1, U_1), \dots, (G_k, U_k)$ such that (1) $G = G_0$ and $U_0 = U$; (2) each U_i induces a strong end of G_i ; (3) G_{i+1} is obtained from G_i by contracting U_i , i.e., $G_{i+1} = G_i/U_i$; (4) G_k is factor-critical and U_k is empty (see an example in Fig. 5.6). From the definition, clearly, each $G_i[U_i]$ has a 1-factor.

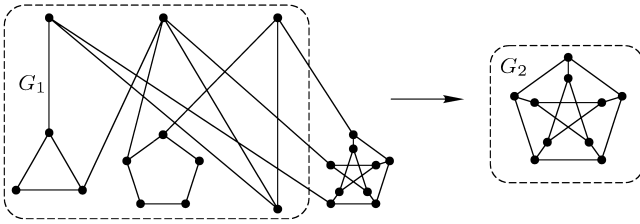


Fig. 5.6 An example of end decomposition.

Theorem 5.3.14 (Frank [214]). *Every graph has an end decomposition $(G_0, U_0), (G_1, U_1), \dots, (G_k, U_k)$. Moreover, the length k of an arbitrary end decomposition depends only on G and $\varphi(G) = k$.*

In the same paper, Frank (1993) also provided an algorithm of the complexity $O(|V||E|)$ to construct an end decomposition.

To obtain an explicit formula for $\varphi(G)$, Frank used another graph theory parameter to express it and thus obtained a max-min theorem.

An edge subset J of $E(G)$ is called a *join* if $|C \cap J| \leq |C|/2$ holds for every cycle C of G . For example, any matching is a join. We denote the maximum cardinality of a join of G by $\eta(G)$.

Theorem 5.3.15 (Frank [214]). *For any graph $G = (V, E)$, then*

$$\varphi(G) \geq 2\eta(G) - |V| + 1. \quad (5.3)$$

Proof. We use the ear decomposition and induction to show this. At first, we need a result on how $\varphi(G)$ changes when an ear is removed.

Claim 1. Let $G = G' + P$, where P is an ear of length p . Then $\eta(G') \geq \eta(G) - \lfloor p/2 \rfloor$.

Let u and v be the two (not necessarily distinct) end-vertices of P . Let J be an optimal join of G and p^- (p^+) denote the number of edges of P in J (not in J). Then the restriction J' of J on G' is also a join.

If $p^- \leq p^+$, then $p^- \leq \lfloor p/2 \rfloor$ and we have $\eta(G') \geq \eta(G) - p^- \geq \eta(G) - \lfloor p/2 \rfloor$, we are done. Next, suppose that $p^- > p^+$, that is, $h(P) = p^- - p^+ > 0$. Since J is a join, every path in G' connecting u and v has at least $h(P)$ more edges in J than that not in J . Let P' be a path of G' connecting u and v and with the minimum $h(P')$. Interchanging the edges on P' and denote the resulting set by J'' , i.e., $J'' = J' \triangle P'$. Then J'' is a join of G' . Otherwise, let C' be a cycle of G' such that $|C' \cap J''| > |C'|/2$. Then $P'' = P' \triangle C'$ is a path connecting u and v but $h(P'') < h(P')$, a contradiction to the minimality of P' . For join J'' of G' , we have $|J''| \geq \eta(G) - p^- + h(P)$. Therefore, $\eta(G') \geq \eta(G) - p^- + h(P) = \eta(G) - p^+ \geq \eta(G) - \lfloor p/2 \rfloor$.

For the given graph G , it has an ear decomposition with $m = |E| - |V| + 1$ ears and with $\varphi(G)$ even ears. Let P be the last ear of the decomposition and $\varepsilon = 1$ if P is odd and $\varepsilon = 0$ if P is even.

We use induction on m to show (5.3). Using Claim 1 and the induction hypothesis, we have

$$\begin{aligned} \varphi(G) &= \varphi(G') + 1 - \varepsilon \\ &\geq 2\eta(G') - |V'| + 1 - \varepsilon + 1 \\ &\geq 2(\eta(G) - \lfloor p/2 \rfloor) - |V'| - \varepsilon + 1 + 2 \\ &= 2\eta(G) - (p - 1 + |V'|) + 1 \\ &= 2\eta(G') - |V| + 1. \end{aligned}$$

We complete the proof. □

Corollary 5.3.1. *If G is factor-critical, then any near-perfect matching is a maximum join.*

Proof. If G is critical, then $\varphi(G) = 0$ and by Theorem 5.3.14 $\eta(G) \leq (|V| - 1)/2$. Moreover any near-perfect matching M of G is a join of cardinality $(|V| - 1)/2$ and hence $\eta(G) = (|V| - 1)/2$, that is, M is a maximum join. □

Actually, Frank [214] showed the reverse of Theorem 5.3.14 is also true but the proof is rather involved and we omit it here. He established the following min-max theorem between $\eta(G)$ and $\varphi(G)$. Interestingly, any end decomposition determines an optimal critical-making set and an optimal join which can be constructed as the union of 1-factors in the building blocks.

Theorem 5.3.16 (Frank [214]). *For any connected graph G , then $\varphi(G) = 2\eta(G) - |V| + 1$. That is,*

$$\min\{|F| \mid F \text{ is a critical-making set}\} = \max\{|J| \mid J \text{ is a join}\}.$$

Moreover, let $(G_0, U_0), (G_1, U_1), \dots, (G_k, U_k)$ be an end decomposition. Then

- (a) let f_i be an arbitrary edge of G_i ($i = 0, \dots, k-1$) connecting X_i and $V(G_i) - X_i$ (where X_i is the strong barrier belonging to $G_i(U_i)$). Then $\{f_0, f_1, \dots, f_{k-1}\}$ is a minimum critical-making set of G ;
- (b) let N_i be any 1-factor of $G_i(U_i)$ ($i = 0, \dots, k-1$) and M_k any near-perfect matching of G_k . Then $J = N_0 \cup \dots \cup N_{k-1} \cup M_k$ is a maximum join of G .

In Section 5.1, we pointed out that bicritical decomposition can be pushed a step further to bricks, then it can be shown that a list of bricks in the new decomposition is an invariant of the graph G and moreover the list of bricks is uniquely determined.

Let G be a 1-extendable graph but not bicritical. Now we describe the procedure of brick decomposition. Note that the first phase is same as (C1) and (C2) in bicritical decomposition in Section 5.1.

Phase I. Let S be a barrier with $|S| = k \geq 2$, and let $H_1, \dots, H_k$ be the components of $G - S$. Let G_i be the graph obtained from G by shrinking all the vertices of $V(G) - V(H_i)$ to a single vertex b_i . Let G_S denote the bipartite graph obtained from G by shrinking each $V(H_i)$ to a single vertex h_i (see an illustration in Fig. 5.7). We know that the graphs $G_S, G_1, \dots, G_k$ are also 1-extendable ([401]).

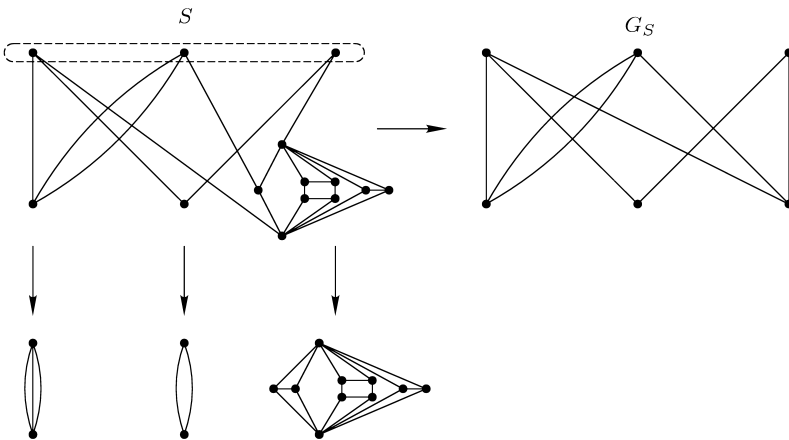


Fig. 5.7 An example of brick decomposition in Phase I.

Let us ignore the graph G_S , and also all graphs which have only two vertices, and all the bicritical graphs. Repeat the procedure Phase I with the non-bicritical block G_i . Eventually, we end up with a list of bicritical graphs. Recall that every bicritical graph is 2-connected and 3-edge-connected. Next step, we turn the bicritical blocks into bricks by decomposing along a 2-cutset as follows.

Phase II. Let G be a bicritical graph but not a brick. Let $\{u, v\}$ be a 2-cutset. Then G can be written as the union of two edge-disjoint subgraphs G' and G'' and $V(G') \cap V(G'') = \{u, v\}$. Add an edge joining u and v to G' and G'' and denote the new graphs by G_1 and G_2 , respectively. Then G_1 and G_2 are bicritical again ([410]). We can continue this procedure on each bicritical block until a list of bricks is obtained (see an illustration in Fig. 5.8).

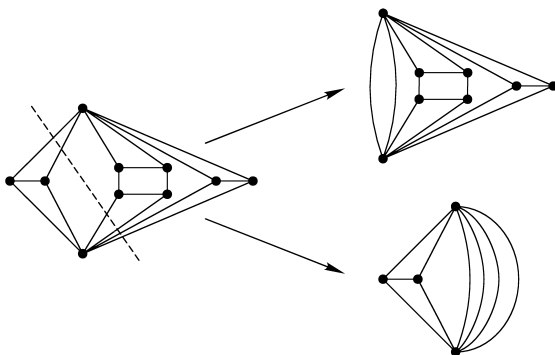


Fig. 5.8 An example of brick decomposition in Phase II.

We call this procedure the *brick decomposition* of G . Note that there are several arbitrary choices made during this procedure: the choice of the barriers in Phase I, and then also the choice of the 2-cutsets in Phase II. However, Lovász (1987) proved that nevertheless the result of the procedure is essentially uniquely determined.

Theorem 5.3.17 (Lovász [408]). *The list of bricks obtained in the brick decomposition procedure is uniquely determined up to isomorphism and the multiplicity of edges. In particular, the number of bricks in the decomposition is invariant of G .*

For each 1-factor M in a graph G , we associate a binary vector of length $|E(G)|$ with M , where the i th entry is 1 if edge i of G belongs to M and is 0, if it does not. We call the linear span of all such binary vectors over the real number *perfect matching polytope*, denote by $\mathcal{PM}(G)$. Edmonds, Lovász and Pulleyblank [167] showed an intimate connection between brick decomposition and the dimension of $\mathcal{PM}(G)$.

Theorem 5.3.18 (Edmonds, Lovász and Pulleyblank [167]). *If G is a 1-extendable graph, then*

$$(\text{the dimension of } \mathcal{PM}(G)) = |E(G)| - |V(G)| + 1 - b(G),$$

where $b(G)$ is the number of bricks in brick decomposition of G .

After the dimension of $\mathcal{PM}(G)$ was determined, Edmonds, Lovász and Pulleyblank described which defining inequalities give facets of $\mathcal{PM}(G)$, and which of them yield the same facet. More details on this topic can be found in [412].

5.4 Bricks and optimal ear decomposition

Recently, great progresses were made by various authors (e.g., Campos, de Carvalho, Cheriyan, Little, Lucchesiy and Murty [111; 113? ; 114; 115? ; 116; 118; 119; 117; 120; 121; 122; 124; 123], McCuaig [430; 431], Norine and Thomas [467; 468]) on bricks, braces and their decomposition theory. In this section, we give a brief review of the recent advances. All graphs in this section are assumed to be 1-extendable unless otherwise stated.

Recall that a brick is a 3-connected bicritical graph. Note that bicritical graphs must be non-bipartite. For bipartite graphs, we modify the definition slightly. A *brace* is a connected bipartite graph such that deleting any two vertices from each color class, the remaining bipartite graph has a 1-factor. Clearly, every brace is 1-extendable. It is not hard to see that a bipartite graph $G = (U, W)$ is a brace if and only if for each subset $X \subset U$ with $0 < |X| < |U|$ has at least $|X| + 2$ neighbors in W .

There are three special bricks, the Petersen graph, prism and R_8 (see Fig. 5.9), which are excluded from many known results and we list here first.

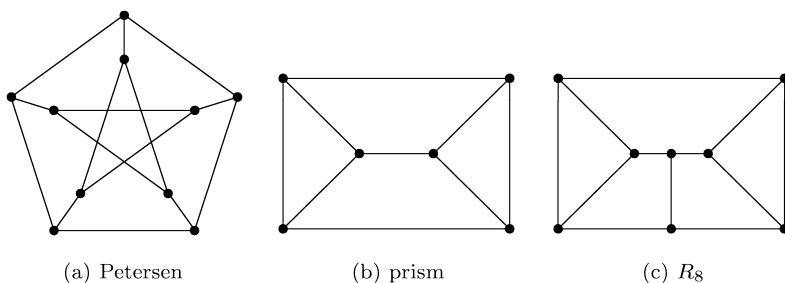


Fig. 5.9 Three special bricks.

Let G be a graph, denote by $\nabla(X)$ the set of edges with one end in X and the other in $V(G) - X$. A *cut* in G is any set of the form $\nabla(X)$ for some $X \subset V(G)$. We call $\nabla(X)$ the cut determined by X . The sets X and $V(G) - X$ are called the *shores* of the cut.

Let G be a 1-extendable graph, a cut C is *tight* if $|C \cap M| = 1$ for every 1-factor M of G . Clearly, for any 1-factor M , and any vertex $v \in V(G)$, $|\nabla(v) \cap M| = 1$. Such cuts are called *trivial*, and the other tight cuts are *nontrivial*. Let us examine a few special cases.

Let G be a 1-extendable bipartite graph with bipartition (U, W) , but not a brace. Then by Hall's Theorem there is a set $X \subset U$ with $0 < |X| \leq |U| - 2$ such that $|N(X)| = |X| + 1$. Then $\nabla(X \cup N(X))$ is a nontrivial tight cut. Indeed, a nontrivial tight cut in a bipartite 1-extendable graph is of this form (Exercise!).

Next, assume that G is a 1-extendable non-bipartite graph but not bicritical. Then there exist distinct vertices $u, v \in V(G)$ such that $G - \{u, v\}$ has no 1-factors, by Tutte's 1-Factor Theorem, there exists a nontrivial barrier X such that $o(G - X) = |X|$. Since G is 1-extendable, no edge of G has both ends in X . As G is not bipartite, some component of $G - X$, say H , has more than one vertex. Then $\nabla(V(H))$ is a nontrivial tight cut. We call this tight cut *barrier cut*.

Finally, if G is bicritical but not 3-connected, then there exist distinct vertices u, v such that $G - \{u, v\}$ is disconnected. As $\{u, v\}$ is not a barrier, then all the components of $G - \{u, v\}$ are even. Let A be the vertex set of a component of $G - \{u, v\}$ and let B be the vertex set of the union of all the remaining components. Note that $|A|$ is even, then $\nabla(A \cup \{u\})$ and $\nabla(A \cup \{v\})$ are nontrivial tight cuts. We call such tight cuts *2-separation cuts*.

We see three types of nontrivial tight cuts from the above discussion. Are there more types out there? Does every nontrivial tight cut arise from ones described above? The next theorem implies that if a graph has a nontrivial tight cut, then such a cut can be arisen in one of the ways described in the previous paragraph. This fact can be stated as an innocent-looking but nontrivial result, which is the main result of [167]. Note that until recently, the only proof was based on LP duality. Szegedi [543] obtained a simple and direct proof.

Theorem 5.4.1 (Tight Cut Lemma, Edmonds, Lovász and Pulleyblank [167]). *A brick has no nontrivial tight cuts.*

In other words, if a graph has any non-trivial tight cut, then it has either a non-trivial barrier cut or a 2-separation cut. It leads us to the next important theorem.

Theorem 5.4.2 (Lovász [408]). *Let G be a 1-extendable graph, then G has no nontrivial tight cuts if and only if G is a brick or a brace.*

Theorem 5.4.2 suggests another procedure which can decompose any 1-extendable graph into “smaller building blocks” — bricks and braces. This is so-called “tight cut decomposition”, which is due to Lovász [408] and is a further extension of brick decompositions.

Let $C := \nabla(X)$ be a nontrivial tight cut in a 1-extendable graph G , let G_1 be a graph obtained from G by contracting X into a single vertex and

deleting all resulting parallel edges, and let G_2 be defined analogously by contracting $V(G) - X$. We call G_1 and G_2 the two C -contractions of G . We shall see in the next theorem that both G_1 and G_2 are 1-extendable. But the converse may not be sure, for example. This enables us to work on smaller graphs G_1 and G_2 when dealing with problems related to matchings.

Theorem 5.4.3. *Let G be a 1-extendable graph, and let C be a nontrivial tight cut of G . Then the two C -contractions G_1 and G_2 of G are also 1-extendable.*

Tight Cut Decomposition:

Given a 1-extendable graph G , if every tight cut is trivial, then G is a brick or brace and we stop; otherwise, look for a nontrivial tight cut C . Let G_1 and G_2 be the two C -contractions of G . Then G_1 and G_2 are also 1-extendable. If G_1 and/or G_2 has a nontrivial tight cut then we repeat this procedure. So finally we end up with a list of graphs which do not have nontrivial tight cut, i.e., bricks and braces. We call this procedure *tight cut decomposition* and the list of bricks and braces obtained, the *result* of this procedure.

Lovász proved that the list of indecomposable graphs obtained at the end of the procedure does not depend on the choices of tight cuts made during the process.

Theorem 5.4.4 (Lovász [408]). *The results of any two applications of the tight cut decomposition procedure on a 1-extendable graph G are the same list of bricks and braces, except possibly for the multiplicities of edges.*

To prove Theorem 5.4.4, we need the following simple property of tight cuts which was proved by Edmonds, Lovász and Pulleyblank [167].

Lemma 5.4.1. *Let G be a 1-extendable graph, and $\nabla(X)$ and $\nabla(Y)$ be tight cuts such that $|X \cap Y|$ is odd. Then $\nabla(X \cap Y)$ and $\nabla(X \cup Y)$ are also tight cuts of G . Moreover, no edges link $X - Y$ and $Y - X$.*

Two cuts $\nabla(X)$ and $\nabla(Y)$ are said *crossed* if $X \not\subset Y$, $Y \not\subset X$, $X \cap Y \neq \emptyset$, and $X \cup Y \neq V$.

Let G be a 1-extendable graph. We break G into two subgraphs G_1 and G_2 along a tight cut $C = V(S)$, where $S \subset V(G_1)$. Let C' be a tight cut in G_1 , then C' can be written as $\nabla(T)$ for some $T \subset S$. So T determines a cut in G which is uncrossed with C . It is not hard to see that this cut must be tight in G . Conversely, if C' is a tight cut in G which is uncrossed with C , say $C' = \nabla(T)$ for some $T \subset S$, then T determines a tight cut in G_1 . Hence all cuts used in a tight cut decomposition procedure correspond to distinct mutually uncrossed nontrivial tight cuts in the original graph G , and they must form an (inclusionwise) maximal family with these properties. Conversely, if we have any maximal family $\mathcal{F}$ of distinct mutually uncrossed

nontrivial tight odd cuts in G , then we can use them in arbitrary order to break up the graph G into $|\mathcal{F}| + 1$ 1-extendable graphs, and these graphs do not have nontrivial tight cuts by the maximality of $\mathcal{F}$. So we have a tight cut decomposition procedure. It is also clear that the resulting list of bricks and braces is independent of the order in which the cuts are used. So a tight cut decomposition procedure corresponds to a maximal family of mutually uncrossed nontrivial tight cuts and vice versa.

Proof of Theorem 5.4.4. We use induction on $|V(G)|$. Let us consider any two tight cut decomposition procedures and let $\mathcal{F}$ and $\mathcal{F}'$ be the two corresponding families of tight cuts.

Case 1. $\mathcal{F}$ and $\mathcal{F}'$ have a common member C . Then we can start both $\mathcal{F}$ and $\mathcal{F}'$ with breaking along this cut C . This results in the same two graphs G_1 and G_2 in both cases, and so both decompositions produce the union of the result of a tight cut decomposition of G_1 , and one of G_2 . Since by the induction hypothesis these lists depend on G_1 and G_2 , only, this proves the assertion.

Case 2. There are cuts $C \in \mathcal{F}$ and $C' \in \mathcal{F}'$ which are uncrossed. Let $\mathcal{F}''$ be any maximal set of mutually uncrossed nontrivial tight cuts containing C and C' . Then by Case 1, the decomposition procedures associated with $\mathcal{F}$ and $\mathcal{F}''$, as well as the procedures associated with $\mathcal{F}$ and $\mathcal{F}'$, end up with the same lists of bricks and braces. Hence $\mathcal{F}$ and $\mathcal{F}'$ end up with the same list.

Case 3. There are cuts $C = \nabla(T) \in \mathcal{F}$ and $C' = \nabla(T') \in \mathcal{F}'$ such that $|T \cap T'|$ is odd and at least three. From Lemma 5.4.1, $C'' = \nabla(T \cap T')$ is also a tight cut, which is clearly nontrivial. Let $\mathcal{F}''$ be a maximal family of mutually uncrossed nontrivial tight cuts containing C'' . Then C'' is uncrossed with C , and hence the decompositions associated with $\mathcal{F}$ and $\mathcal{F}''$ result in the same list of bricks and braces, and similarly for $\mathcal{F}'$ and $\mathcal{F}''$. This proves the assertion.

To complete the proof, consider any $C \in \mathcal{F}$ and $C' \in \mathcal{F}'$. If C and C' are the same, we have done according to Case 1. If they are uncrossed, we are done by Case 2, so assume that they are crossed. Choose shores T of C and T' of C' such that $|T \cap T'|$ is odd (this is clearly possible). If $T \cap T'$ is not a singleton, then we are done by Case 3, so assume that $T \cap T' = \{u\}$ for some vertex u . If $V(G) - T - T'$ is not a singleton, then again we can apply Case 3 by replacing T and T' by their complements. So assume that $V(G) - T - T' = \{v\}$ for some vertex v .

By Lemma 5.4.1, no edge connects $T - T'$ to $T' - T$, then the pair $\{u, v\}$ separates the graph G . Hence C and C' are two 2-separation cuts defined by the same separating pair of vertices. If $\mathcal{F} = \{C\}$ and $\mathcal{F}' = \{C'\}$, then it is

obvious that breaking G along C and along C' results in isomorphic graphs (after deleting multiple edges), and we are done. So assume that, e.g., $\mathcal{F}$ contains another cut $C_0 = \nabla(S)$. Then C_0 is uncrossed with C , and so we may assume, without loss of generality, that $S \cap T = \emptyset$. If C_0 is uncrossed with C' , then we are again finished by Case 2; so assume that this does not happen. Then S is not a subset of T' and hence we must have $v \in S$ and so $|S \cap (V(G) - T')| = 1$. Hence trivially $|(V(G) - S) \cap T'|$ is odd. But then we are finished by Case 3 since $|(V(G) - S) \cap T'| = |T' - S| \neq 1$, as $C_0 \neq C$. $\square$

From this theorem, the numbers of bricks and braces resulting from a tight cut decomposition of a 1-extendable graph G are independent of the tight cut decomposition process. We denote $b(G)$ the number of bricks of G in a tight cut decomposition and thus $b(G)$ is an invariant of G . The number of bricks of G whose underlying simple graphs are Petersen graphs is also an invariant of G , we denote it by $p(G)$. Clearly, $b(G) + p(G)$ is also an invariant of G . The numbers $b(G)$ and $(b+p)(G)$ play important roles in arguments of many results related to 1-extendable graphs.

In particular, $b(G) = 0$ if and only if G is bipartite, and $b(G) = 1$ if and only if for every tight cut C of G , one of the C -contractions is bipartite and the other has exactly one brick. We refer a 1-extendable graph with $b(G) = 1$ as a *near-brick*.

Vazirani and Yannakakis (1989) used the tight cut decomposition procedure to reduce the study of Pfaffian graphs to bricks and braces. Recall that an orientation of a graph G is *Pfaffian*, if for every even cycle C , $G - V(C)$ has a 1-factor such that an odd number of edges directed in either direction of the cycle. The significance of Pfaffian orientations is that if a graph has such an orientation, then the number of 1-factors, $\Phi(G)$, can be computed in polynomial time.

Theorem 5.4.5 (Vazirani and Yannakakis [567]). *A graph G is Pfaffian if and only if every brick and brace of G is Pfaffian.*

This theorem highlighted the importance of bricks and braces, besides its importance in determining the dimensions of matching polytopes and matching lattices.

Next, we generalize the concept of tight cuts and introduce “separating cuts”, which is useful to refine the properties of tight cuts. For a 1-extendable graph G , a cut $C = \nabla(X)$ is a *separating cut* of G if both the C -contractions of G are 1-extendable. Clearly, from Theorem 5.4.2, every tight cut is a separating cut, but the converse is not true. For example, the Petersen graph, for any 5-cycle C , the cut $\nabla(V(C))$ is separating but not tight. The following theorem provides a useful characterization of separating cuts.

Theorem 5.4.6 (de Carvalho, Lucchesi and Murty [118]). *Let G be a 1-extendable graph. A cut C is separating if and only if for each edge*

$e \in E(G)$, there exists a 1-factor M_e containing e such that $|C \cap M_e| = 1$. Moreover, we have

- (a) every tight cut is a separating cut;
- (b) a cut in a bipartite 1-extendable graph is tight if and only if it is separating.

For bricks, Reed and Wakabayashi (2003) obtained the following characterization for separating cuts.

Theorem 5.4.7. *A brick G has a separating cut if and only if it has two disjoint odd cycles C_1 and C_2 such that $G - (V(C_1) \cup V(C_2))$ has a 1-factor.*

Next, we introduce a new graphic parameter involved separating cuts. Denote $\mathcal{M}_G$ (or $\mathcal{M}$ if there is no confusion arisen) the set of 1-factors of G . For each odd cut C and each positive odd integer i , we define $\mathcal{M}_i(C)$ as $\mathcal{M}_i(C) = \{M \in \mathcal{M} \mid |M \cap C| = i\}$. So $\mathcal{M}_i(C) = \emptyset$ for $i \geq 3$ if C is tight. For each separating cut C of G , the characteristic $\lambda(C)$ of C is defined as follow:

$$\lambda(C) := \begin{cases} \min\{i > 1 \mid \mathcal{M}_i(C) \neq \emptyset\} & \text{if } C \text{ is not tight;} \\ \infty & \text{otherwise.} \end{cases}$$

The characteristic $\lambda(G)$ of a 1-extendable graph G is defined as follows:

$$\lambda(G) := \min\{\lambda(C) \mid C \text{ is a separating cut of } G\},$$

where the minimum is taken over all the separating cuts of G .

For example, $\lambda(\overline{C}_6) = 3$ and $\lambda(\text{Petersen graph}) = 5$. For this new parameter, we have following two interesting results. More applications of $\lambda(G)$ are given later in this section.

Theorem 5.4.8 (de Carvalho (1997)). *If G is a brick, then $\lambda(G) \in \{3, 5, \infty\}$. Moreover, if $\lambda(G) = 5$, then the underlying simple graph G is the Petersen graph.*

Theorem 5.4.9 (de Carvalho, Lucchesi and Murty [118]). *The characteristic of a 1-extendable graph is the minimum of the characteristics of its bricks and braces.*

Now we return to ear decompositions of a graph. Recall that every 1-extendable graph G has an ear decomposition of G : $G_0 \subset G_1 \subset \dots \subset G_m = G$ in which each step at most two ears are added. In trying to establish the existence of ear decomposition with given properties, it is often convenient to find the subgraphs in the ear decomposition in the reverse order starting with $G_m = G$. Thus, after obtaining a subgraph G_i in the sequence which is different from K_2 , we find a suitable removable ear (single or double) and obtain G_{i-1} from G_i by removing that ear from G_i . For example, to show that a 1-extendable graph G has an ear decomposition, it suffices to show that

every 1-extendable graph different from K_2 has a removable (single or double) ear. A single ear P of G is *removable* if the graph $G - P$ is 1-extendable. (A removable ear of length one is called a removable edge, or a *removable singleton*.) The notion of a removable double ear (P_1, P_2) in G is similarly defined and a removable double edges is called a *removable doubleton*. (In this case, it is to be understood that neither P_1 nor P_2 is a removable single ear.) A removable ear in G is either a single or double ear which is removable.

The first result in this direction is due to Lovász [408] that every brick different from K_4 and prism (i.e., $\overline{C}_6$) has a removable edge. de Carvalho, Lucchesi and Murty (1999) generalized this result and prove the following.

Theorem 5.4.10 (de Carvalho, Lucchesi and Murty [116]). *Every brick different from K_4 and $\overline{C}_6$ has $(\Delta - 2)$ removable edges, where Δ is the maximum degree of G .*

The next interesting result addresses the effect of removing a double ear.

Theorem 5.4.11 (Lovász [408]). *If G is a brick and R is a removable double ear in G , then $G - R$ is bipartite.*

In order to prove a given statement using inductive arguments, it is convenient to investigate removable ears. For example, if we attempt to find an ear decomposition of a 1-extendable graph G with at most d double ears, it is necessary to show that G has a removable ear R such that the graph $G - R$ has an ear decomposition with at most d double ears if R is a single ear, or $d - 1$ double ears, if R is a double ear. This is the motivation for the notions of rank of an ear, and of $(b + p)$ -removable ear.

The *rank* of an ear R of G , denoted by $r(R)$, is defined to be 0 if R is a singleton and 1 if R is a doubleton. A removable ear R of a graph G is *b-removable* in G if $b(G - R) = b(G) - r(R)$. A removable ear R of G is *$(b + p)$ -removable* in G if $(b + p)(G - R) = (b + p)(G) - r(R)$. de Carvalho *et al.* [118; 119] proved the following important theorem.

Theorem 5.4.12 (de Carvalho, Lucchesi and Murty [118; 119]). *Every brick G has two edge-disjoint $(b + p)$ -removable ears. Moreover,*

- (a) *if $\lambda(G) = \infty$ and $G \neq K_4$, then G has at least three edge-disjoint $(b + p)$ -removable ears, of which at least two are singletons. Brick K_4 has three removable doubletons;*
- (b) *if $3 < \lambda(G) < \infty$, then the underlying simple graph of G is the Petersen graph and $\lambda(G) = 5$. Moreover, every edge of G is $(b + p)$ -removable in G ;*
- (c) *if $\lambda(G) = 3$ and $G \neq \overline{C}_6$, $G \neq R_8$, then G has two $(b + p)$ -removable edges. Graph $\overline{C}_6$ has three removable doubletons and no removable edge. Graph R_8 has one $(b + p)$ -removable edge and two removable doubletons.*

Next, we go on to study the effect of removal of the removable edges/ears on the parameters $\lambda(G)$ and $b(G)$.

Theorem 5.4.13 (de Carvalho, Lucchesi and Murty [118]). *Let G be a 1-extendable graph and let e be a removable edge of G . Then $b(G - e) \geq b(G)$.*

Proof. By induction on the number of edges.

If G is bipartite, in particular, G is a brace, then $b(G - e) = 0 = b(G)$. If G is a brick, then $G - e$ can not be bipartite, and so $b(G - e) \geq 1 = b(G)$. So we assume that G is neither a brace nor a brick. From Theorem 5.4.2, G has a nontrivial tight cut, say C , and let the two C -contractions of G be G_1 and G_2 . Obviously, $G_1 - e$ and $G_2 - e$ are the two $(C - e)$ -contractions of $G - e$. Moreover, $C - e$ is a nontrivial tight cut of $G - e$. By the induction of hypothesis,

$$b(G - e) = b(G_1 - e) + b(G_2 - e) \geq b(G_1) + b(G_2) = b(G),$$

and the assertion holds. $\square$

Theorem 5.4.14 (de Carvalho, Lucchesi and Murty [118]). *Let G be a 1-extendable graph and let e and R be a removable edge and a removable doubleton, respectively. Then*

- (a) $\lambda(G - e) \geq \lambda(G)$;
- (b) $(b + p)(G - e) \geq (b + p)(G)$;
- (c) $(b + p)(G - R) = (b + p)(G) - 1$.

Theorem 5.4.15 (de Carvalho, Lucchesi and Murty [117]). *Every 1-extendable graph distinct from K_2 and C_{2n} ($n \geq 2$) has two edge-disjoint $(b + p)$ -removable ears.*

The following statement is an enriched version of Two Ears Theorem, which is accomplished by de Carvalho, Lucchesi and Murty (2002).

Theorem 5.4.16 (Optimal Ear Decomposition Theorem, de Carvalho, Lucchesi and Murty [117]). *Every 1-extendable graph G admits an ear decomposition that uses exactly $(b + p)(G)$ double ears.*

We turn our attention now to the subject of generating bricks and braces. Several articles discussed this topic in great details (e.g., [124; 123; 430; 431; 468]).

At first, we introduce a procedure due to de Carvalho, Lucchesi and Murty [124].

Reduction Operation:

Let G be a graph, and let v_0 be a vertex of G of degree two incident with the edges $e_1 = v_0v_1$ and $e_2 = v_0v_2$. Let H be a graph obtained from G by contracting both e_1 and e_2 and deleting all resulting parallel edges. Then H is obtained from G by *bicontracting* or *bicontracting the vertex v_0* . We call a graph H to be a *reduction* of G if H can be obtained from G by deleting an edge and then bicontracting all new vertices of degree two.

With this process, we are able to generate many smaller bricks from a given brick.

Theorem 5.4.17 (de Carvalho, Lucchesi and Murty [124]). *If G is a brick other than K_4 , $\overline{C}_6$, and the Petersen graph, then a reduction of G is a brick, other than the Petersen graph.*

Given a non-Petersen brick G , the reduction operation can be repeated until we reach K_4 or $\overline{C}_6$. By reversing the process, Theorem 5.4.17 can be viewed as a brick generating procedure.

Theorem 5.4.17 has many interesting applications. First of all, it implies several results regarding various spaces generated by 1-factors, including a deep theorem of Lovász [408] that characterizes the matching lattice of a graph. Second, it implies that the decision problems “Is a given orientation of a graph Pfaffian” and “Is an input graph Pfaffian” are polynomial-time equivalent, which was proved by Vazirani and Yannakakis [567] (more precisely the most difficult part of that theorem, namely that it holds for bricks). Third, it can be used to prove a uniqueness theorem for Pfaffian orientations.

A graph H is a *matching minor* of G if H can be obtained from a nice subgraph of G by repeatedly bicontracting vertices of degree two.

There is a strengthening of Theorem 5.4.17, which we now describe it. First, the starting graph can be any matching minor of G except K_4 and $\overline{C}_6$; second, reduction can be replaced by a more restricted operation: proper reduction. A graph H is a *proper reduction* of a graph G if it is a reduction in such a way that the bicontractions involved do not produce parallel edges. Unfortunately, Theorem 5.4.17 does not hold for proper reductions, but all the exceptions can be conveniently described as follow:

Let C_1 and C_2 be two vertex-disjoint cycles of length $n \geq 3$ with vertex-sets $\{u_1, u_2, \dots, u_n\}$ and $\{v_1, v_2, \dots, v_n\}$ (in order), respectively, and let G_1 be the graph obtained from the union of C_1 and C_2 by adding an edge joining u_i and v_i for each $i = 1, 2, \dots, n$. We call G_1 a *planar ladder*. Let G_2 be the graph consisting of the cycle C_{2n} with vertex-set $\{u_1, u_2, \dots, u_{2n}\}$ and n edges $u_i u_{n+i}$ for $i = 1, 2, \dots, n$. We call G_2 a *Möbius ladder*. A *ladder* is a planar ladder or a Möbius ladder. Let G_3 denote a graph obtained from G_1 by deleting the edge $u_1 u_2$ and contracting the edges $u_1 v_1$ and $u_2 v_2$. We refer G_3 as a *staircase*. Let G_4 be a graph obtained from a path $P = v_1 v_2 \dots v_t$ by adding two new vertices x, y and edges xv_i and yv_j for $i = 1, t$ and all even $i \in \{1, 2, \dots, t\}$ and $j = 1, t$ and all odd $j \in \{1, 2, \dots, t\}$. Let G_5 be obtained from G_4 by adding the edge xy . We call G_5 an *upper prismoid*, and if $t \geq 4$, then we call G_4 a *lower prismoid*. A *prismoid* is a lower prismoid or an upper prismoid. Finally, we denote the wheel with six vertices by G_6 (see Fig. 5.10).

We are now ready to state a strengthening of Theorem 5.4.17, proved by Norine and Thomas (2007).

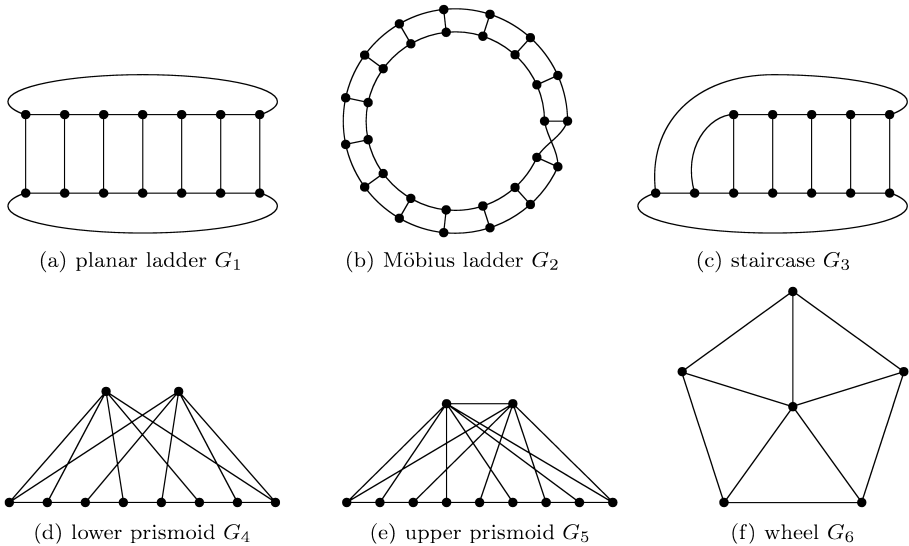


Fig. 5.10 The graphs G_1 – G_6 .

Theorem 5.4.18 (Norine and Thomas [468]). *Let H, G be bricks, where H is a matching minor of G . Assume that $H \neq K_4, \overline{C}_6$, and $G \notin \{G_1, \dots, G_6\}$. Then a graph isomorphic to H can be obtained from G by repeatedly taking proper reductions in such a way that all the intermediate graphs are non-Petersen bricks.*

As a counterpart to Theorem 5.4.18, Norine and Thomas [468] described the starting graphs for the generation process.

Theorem 5.4.19 (Norine and Thomas [468]). *Let G be a brick not isomorphic to $K_4, \overline{C}_6$ or the Petersen graph. Then G has a matching minor isomorphic to one of the following seven graphs: the graph obtained from the prism by adding an edge, the lower prismoid on eight vertices, the staircase on eight vertices, the staircase on ten vertices, the planar ladder on ten vertices, the wheel on six vertices, and the Möbius ladder on eight vertices.*

If H is a matching minor of a brick G and $G \in \{G_1, \dots, G_6\}$, then $H \in \{G_1, \dots, G_6\}$ and it can be obtained from a graph isomorphic to G by taking (improper) reductions in such a way that all intermediate graphs are bricks. Thus Theorems 5.4.18 and 5.4.19 imply Theorem 5.4.17. Theorems 5.4.18 and 5.4.19 are powerful tools used in proving several results about the minimum degree of minimal bricks (see [467]), and to generate interesting examples of Pfaffian bricks.

McCuaig [431] proved an analogue of Theorem 5.4.18 for braces. To state his result we need another exceptional class of graphs. Let C be an even cycle with vertices $v_1, v_2, \dots, v_{2t}$ in order ($t \geq 2$) and let G_7 be a graph obtained

from C by adding two new vertices v_{2t+1} and v_{2t+2} and edges joining v_{2t+1} to the vertices of C with odd indices and v_{2t+2} to the vertices of C with even indices. Let G_8 be obtained from G_7 by adding an edge $v_{2t+1}v_{2t+2}$. We call G_8 an *upper biwheel*, and if $t \geq 3$ we refer G_7 as a *lower biwheel*. A *biwheel* is a lower biwheel or an upper biwheel. McCuaig's brace generating theorem can be stated as follows.

Theorem 5.4.20 (McCuaig [431]). *Let H and G be braces, where H is a matching minor of G . Assume that if H is a planar ladder, then it is the largest planar ladder matching minor of G ; similarly, for Möius ladders, lower biwheels and upper biwheels. Then H can be obtained from G by repeatedly taking proper reductions in such a way that all the intermediate graphs are braces.*

Our final topic in this section is matching lattice, in particular, an elegant characterization of the dimension of this space due to Lovász [408]. We start with the concept of matching lattices, and then discuss the basis and dimension of a matching lattice.

The reader is reminded again that the graph G discussed here is assumed to be 1-extendable, unless otherwise stated.

Recall that $\mathcal{M}_G$ is the set of 1-factors of a graph G . For a subset $A \subseteq E$, χ^A denotes the incidence vector of A . The *matching lattice* of G , $\text{Lat}(G)$, is the set of all *integer* linear combinations of vectors in $\{\chi^M \mid M \in \mathcal{M}_G\}$.

Let w be a vector in $\mathbb{Z}^E$. In order to determine the necessary and sufficient conditions for w to be in $\text{Lat}(G)$, we begin the analysis of this question by reducing it to a case in which G is free of nontrivial tight cuts.

Theorem 5.4.21 (Lovász [408]). *Let C be a tight cut of a graph G , and G' and G'' be the two C -contractions of G . Let w be a vector in $\mathbb{Z}^E$, and w' and w'' denote the restrictions of w to $E(G')$ and $E(G'')$, respectively. Then, vector w lies in $\text{Lat}(G)$ if and only if vector w' lies in $\text{Lat}(G')$ and vector w'' lies in $\text{Lat}(G'')$.*

Hence, fix a tight cut decomposition of G , then a vector w of $\mathbb{Z}^E$ lies in $\text{Lat}(G)$ if and only if for every brick or brace H obtained by the tight cut decomposition of G , the restriction of w to $E(H)$ lies in $\text{Lat}(H)$. So this theorem can be used as a tool to reduce the characterization of $\text{Lat}(G)$ to the case in which G is free of nontrivial tight cuts.

To find the necessary condition for w belonging to $\text{Lat}(G)$, we consider the *matching space* of G , $\text{Lin}(G)$, which is the linear space over $\mathbb{R}$ generated by $\{\chi^M \mid M \in \mathcal{M}\}$. Clearly, $\text{Lat}(G) \subseteq \text{Lin}(G) \cap \mathbb{Z}^E$. Thus, the necessary conditions for a vector w of $\mathbb{R}^E$ to belong to $\text{Lat}(G)$ is in $\text{Lin}(G)$.

From the definition of tight cuts, then $|M \cap C| = 1$ for any 1-factor M and any tight cut C . Thus if $w = \sum_{M \in \mathcal{M}} \alpha^M \chi^M$, then $w(C) = \sum_{M \in \mathcal{M}} \alpha^M$.

So a further necessary condition for a vector w in $\mathbb{R}^E$ belonging to $\text{Lin}(G)$ is that

$$w(C) = w(D), \text{ for any two tight cuts } C \text{ and } D \text{ of } G. \quad (**)$$

Naddef (1982) proved that these conditions are also sufficient and thus gave a complete characterization of $\text{Lin}(G)$.

Theorem 5.4.22. *A vector w in $\mathbb{R}^E$ is in $\text{Lin}(G)$ if and only if w satisfies (**).*

The above theorem gives a description of a system of linear equations which determines $\text{Lin}(G)$. But, in general, $\text{Lat}(G)$ does not coincide with $\text{Lin}(G) \cap \mathbb{Z}^E$. For example, if P is the Petersen graph, then $\text{Lat}(P)$ is a strict subset of $\text{Lin}(P) \cap \mathbb{Z}^E$. To describe $\text{Lat}(G)$, we need to introduce a new term. Let $\mathcal{C}$ be the set of tight cuts of G . A vector w in $\mathbb{R}^E$ is *regular* over $\mathcal{C}$ if $w(C)$ has the same value for all cuts C in $\mathcal{C}$. Theorem 5.4.22 implies that regularity over the set of all tight cuts of G is a necessary and sufficient condition for w to be in $\text{Lin}(G)$. Further research by Lovász indicated that the regularity condition is also sufficient for braces and bricks to belong to $\text{Lin}(G)$ with only one exception, the Petersen brick. Let Q be a pentagon in the underlying Petersen graph of G . We refer to the subgraph of G induced by $V(Q)$ as a *fattened pentagon* of G .

Now we can state the complete description of $\text{Lat}(G)$ by Lovász (1987).

Theorem 5.4.23 (Lovász [408]). *Let G be a 1-extendable graph and let w be a vector in $\mathbb{Z}^E$. Fix any tight cut decomposition of G . Then, w lies in $\text{Lat}(G)$ if and only if the following conditions hold for each brick or brace H in the given tight cut decomposition*

- (a) *the restriction of w to $E(H)$ is regular, and*
- (b) *if H is a Petersen brick, then, for a specified fattened pentagon Q of H , $w(E(Q))$ is even.*

With the characterization of $\text{Lat}(G)$ above, Lovász obtained an elegant formula for the dimensions of matching lattices.

Theorem 5.4.24 (Lovász [408]). *Let G be a 1-extendable graph. Then*

$$(\text{dimension of } \text{Lat}(G)) = |E(G)| - |V(G)| + 2 - b(G),$$

where $b(G)$ is the number of bricks in a tight cut decomposition.

In fact, the basis of a matching lattice $\text{Lat}(G)$ can consist solely of incidence vectors of 1-factors of G . de Carvalho, Lucchesi and Murty [117] extended the idea on 1-factors associated with an optimal ear decomposition and showed that the corresponding characteristic vectors constitute a basis

for the matching lattice. We brief their idea below.

For a 1-extendable graph G , let $m = |E(G)|$ and $n = |V(G)|$, and let $\mathcal{D} := (G_1, \dots, G_r)$ be an optimal ear decomposition of G , where $r = m - n + 2 - (b + p)$. From Theorem 5.4.24, the dimension of $\text{Lat}(G)$ is $m - n + 2 - b$ and thus the sequence of 1-factors associated with $\mathcal{D}$ is not large enough – it falls short by p 1-factors. So their approach is to supplement the sequence $\mathcal{D}$ with p additional 1-factors so that the set of incidence vectors of 1-factors of the resulting set of $m - n + 2 - b$ 1-factors forms a basis for $\text{Lat}(G)$.

The basic idea is as follows. Suppose that $G_{r-1} = G - R$, where R is a $(b + p)$ -removable ear of G . Further suppose that we have found a set $\mathcal{M}' = \{M'_1, M'_2, \dots, M'_k\}$ of 1-factors of G_{r-1} such that $\mathcal{S}' = \{\chi^{M'_1}, \chi^{M'_2}, \dots, \chi^{M'_k}\}$ is a basis for $\text{Lat}(G_{r-1})$. Let T and U denote, respectively, the sets of odd and even edges (i.e., edges on a path with the respective parity) of R . Then, consider the set $\{M_1, M_2, \dots, M_k\}$ of 1-factors of G , where $M_i = M'_i \cup U$ ($i = 1, 2, \dots, k$). Then, clearly $\{\chi^{M_1}, \chi^{M_2}, \dots, \chi^{M_k}\}$ is linearly independent. This set of course does not generate vectors in $\text{Lat}(G)$ which assume nonzero values on T . If we can find a set $M_{k+1}, \dots, M_{k+n}$ of 1-factors of G , each containing T , such that $\mathcal{S} = \{\chi^{M_1}, \chi^{M_2}, \dots, \chi^{M_k}, \chi^{M_{k+1}}, \dots, \chi^{M_{k+n}}\}$ is a basis of $\text{Lat}(G)$, then we see that the basis $\mathcal{S}'$ of $\text{Lat}(G_{r-1})$ can be extended to a basis $\mathcal{S}$ of $\text{Lat}(G)$ by adding n vectors.

If all vectors w of $\text{Lat}(G)$ with $w(e) = 0$ for all $e \in T$ can be expressed as integer linear combinations of vectors in $\{\chi^{M_1}, \chi^{M_2}, \dots, \chi^{M_k}\}$, then it is easy to extend it to a basis of $\text{Lat}(G)$. Because, in this case, if M_{k+1} is any 1-factor of G containing T , then $\{\chi^{M_1}, \chi^{M_2}, \dots, \chi^{M_k}, \chi^{M_{k+1}}\}$ is a basis of $\text{Lat}(G)$.

It is not always possible to extend a basis of $\text{Lat}(G_{r-1})$ to a basis of $\text{Lat}(G)$ by adding just one vector. However, in general, if G is a 1-extendable graph, and R is a $(b + p)$ -removable ear of G , then a basis of $\text{Lat}(G - R)$ can be extended, in the above sense, to a basis of $\text{Lat}(G)$ by adding either one or two vectors. In fact, if R is also b -removable, then it can be shown that a basis of $\text{Lat}(G - R)$ can be extended to a basis of $\text{Lat}(G)$ by adding the incidence vector of any 1-factor containing T . On the other hand, if R is not b -removable, then a basis of $\text{Lat}(G - R)$ can be extended to a basis of $\text{Lat}(G)$ by adding the incidence vectors of two 1-factors containing T .

Let $\mathcal{M}(\mathcal{D}) = \{M_1, M_2, \dots, M_r\}$ be any set of 1-factors associated with an optimal ear decomposition $\mathcal{D}$. Denote the resulting set of $r + p$ 1-factors obtained by adding p 1-factors to $\mathcal{M}(\mathcal{D})$ described above by $\mathcal{M}^*(\mathcal{D})$. We can state the described procedure as a theorem below. The reader can refer to [117] for the details of its proof.

Theorem 5.4.25 (de Carvalho, Lucchesi and Murty [117]). *Let G be a 1-extendable graph. Let $\mathcal{D}$ be any optimal ear decomposition of G , and $\mathcal{M}^*(\mathcal{D})$ be a set of 1-factors associated with $\mathcal{D}$. Then, the incidence vectors of 1-factors in $\mathcal{M}^*(\mathcal{D})$ is a basis for $\text{Lat}(G)$.*

If every separating cut in a graph G is tight, then we say that G is *solid*. As a conclusion of the section, we include an open problem from de Carvalho, Lucchesi and Murty (2003).

Open Problem 5.4.1 (de Carvalho, Lucchesi and Murty [120]). *Is the decision problem to determine if a given brick is solid in NP.*

Chapter 6

k -Extendable Graphs and n -Factor-Critical Graphs

In 1979, Sumner [538] posed an interesting problem asked to characterize the graphs with the property that *every* matching can be extended to a 1-factor. It turns out that only two classes of graphs, i.e., the complete graphs K_{2n} and the complete bipartite graphs $K_{n,n}$, satisfy this property. Clearly, the property is very strong since we asked that the matchings of *every* size have to extend to a 1-factor. If we relax the property a bit by requiring only the matching with the *same* size to be extended to a 1-factor, then we lead to an interesting refinement, k -extendable graphs, which was introduced by Plummer in 1980.

Let's recall that a graph G is k -extendable if each k -matching of G can be extended to a 1-factor. For convenience, if G has a 1-factor, we refer it 0-extendable. A graph G is said to be n -factor-critical if the subgraph $G - S$ has a 1-factor for any subset S of $V(G)$ with $|S| = n$.

Example 6.0.1. Consider the Petersen graph and the cube in Fig. 6.1. The Petersen graph is 1-extendable graph but not 2-extendable and the cube is 2-extendable.

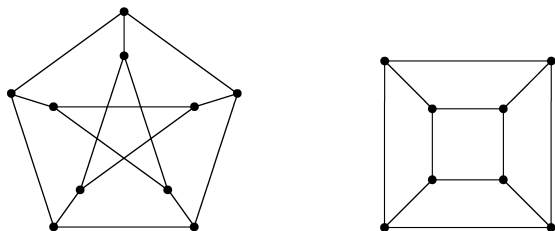


Fig. 6.1 The Petersen graph and the cube.

In this chapter, we mainly concentrate on non-algorithmic aspect of k -extendable graphs and n -factor-critical graphs. However, there remains a fundamental question about the complexity of $ext(G)$. In particular, Frank, Györi and Sebö [217] raised the following question:

Open Problem 6.0.1. *Given a graph G and a positive integer k , does there exist a polynomial algorithm to determine if $\text{ext}(G) = k$?*

For bipartite graphs with a perfect matching, the above problem was solved by Zhang and Zhang [624]. They found an expression of the extendability of bipartite graphs in terms of arc-connectivity of digraphs and thus designed an algorithm to determine the extendability in $O(|V||E|)$ times.

6.1 Characterizations and basic properties

In this section, we study the basic properties of various extensions. In Section 5.1, we studied elementary graphs, 1-extendable graphs and bicritical graphs. They are graphs with lower extendabilities. In this section, we attempt to extend the earlier results to graphs with higher extendabilities. At first, we have the following observation:

Observation 6.1.1. *A graph G is k -extendable if and only if for any matching M of size i ($1 \leq i \leq k$) the graph $G - V(M)$ is $(k - i)$ -extendable.*

There are Tutte-type of characterizations for k -extendable graphs and n -factor-critical graphs. We start with the characterization of k -extendable graphs obtained by Yu (1993), which is a generalization of Theorem 5.1.2.

Theorem 6.1.1 (Yu [616]). *A graph G is k -extendable ($k \geq 1$) if and only if for any $S \subseteq V(G)$ we have*

- (a) $o(G - S) \leq |S|$ and
- (b) $o(G - S) = |S| - 2i$ ($0 \leq i \leq k - 1$) implies that $F(S) \leq i$, where $F(S)$ is the size of a maximum matching in $G[S]$.

Proof. Suppose G is k -extendable. Since G has a 1-factor, (a) follows from Tutte's theorem. Suppose $o(G - S) = |S| - 2i$ ($0 \leq i \leq k - 1$) for some vertex-set $S \subseteq V(G)$. We consider the case that $i = k - 1$ first. In this case, assume $F(S) > k - 1$. Let $e_j = x_j y_j$ ($1 \leq j \leq k - 1$) be $k - 1$ independent edges in $G[S]$. By Observation 6.1.1, $G - \{x_1, y_1, \dots, x_{k-1}, y_{k-1}\}$ is 1-extendable. Let $G' = G - \{x_1, y_1, \dots, x_{k-1}, y_{k-1}\}$ and $S' = S - \{x_1, y_1, \dots, x_{k-1}, y_{k-1}\}$. Then $o(G' - S') = o(G - S) = |S| - 2(k - 1) = |S'|$ and thus any 1-factor of G' must match each vertex of S' to one vertex in a different odd component. Hence S' is an independent set. Thus $F(S) = F(S') + (k - 1) = k - 1 = i$, a contradiction. Since i -extendability implies $(i - 1)$ -extendability (see Theorem 6.1.11), (b) holds for $0 \leq i \leq k - 2$.

Next suppose (a) and (b) hold. We use induction on k to show that G is k -extendable.

For $k = 1$, if G is not 1-extendable, then there exists an edge $e = xy$ so that no 1-factor containing e or $G - \{x, y\}$ has no 1-factors. By Tutte's

1-Factor Theorem, there exists a subset $S' \subseteq V(G - \{x, y\})$ so that $o((G - \{x, y\}) - S') > |S'|$. Let $S = S' \cup \{x, y\}$. Since G has 1-factors, using parity argument we see $|S| \geq o(G - S) = o((G - \{x, y\}) - S') \geq |S'| + 2 = |S|$ and thus $o(G - S) = |S|$. But S contains at least one edge or $F(S) \geq 1$, a contradiction to (b).

Suppose that the claim holds for $k < r$. Consider $k = r$. By the induction hypothesis, (a) and (b) imply that G is $(r-1)$ -extendable. If G is r -extendable, we are done. Otherwise, there exist $r-1$ independent edges $e_i = x_i y_i$ ($1 \leq i \leq r-1$) so that $G' = G - \{x_1, y_1, \dots, x_{r-1}, y_{r-1}\}$ is not 1-extendable. Since G' has a 1-factor, then there exists a set $S' \subseteq V(G')$ so that $o(G' - S') = |S'|$ and $F(S') \geq 1$. Let $S = S' \cup \{x_1, y_1, \dots, x_{r-1}, y_{r-1}\}$. Then $o(G - S) = o(G' - S') = |S'| = |S| - 2(r-1)$ and $F(S) = F(S') + (r-1) \geq r$, which contradicts to the condition (b). $\square$

An alternative characterization of k -extendable graphs similar to Theorem 6.1.1 was given by Chen (1995).

Theorem 6.1.2 (Chen [136]). *A graph G is k -extendable ($k \geq 1$) if and only if $o(G - S) \leq |S| - 2k$ for any $S \subseteq V(G)$ such that $G[S]$ contains k independent edges.*

Aldred, Holton, Lou and Zhong (2004) provided another characterization for k -extendable graphs in terms of alternating paths.

Theorem 6.1.3 (Aldred *et al.* [17]). *Let G be a graph with a 1-factor M . Then G is k -extendable if and only if for each r with $0 \leq r \leq k$ and each $F \subseteq M$ with $|F| = r$, and for any $k-r$ pairs of M -alternating paths $x_i x'_i y'_i y_i$ in $G - V(F)$ such that $x_i x'_i, y_i y'_i \in M$ ($1 \leq i \leq k-r$) and $x_1, y_1, x_2, y_2, \dots, x_{k-r}, y_{k-r}$ are $2(k-r)$ different vertices, there exist m independent M -alternating paths $P_1, P_2, \dots, P_m$ in $G - V(F)$ joining the vertices in $Z = \{x_1, y_1, \dots, x_{k-r}, y_{k-r}\} - \{x'_1, y'_1, \dots, x'_{k-r}, y'_{k-r}\}$, where $|Z| = 2m$, which start and end with edges in $E(G) - M$.*

The following characterization of n -factor-critical graphs due to Yu and Favaron, independently.

Theorem 6.1.4 (Yu [616]; Favaron [202]). *A graph G is n -factor-critical if and only if $|V(G)| \equiv n \pmod{2}$ and for any vertex-set $S \subseteq V(G)$ with $|S| \geq n$, $o(G - S) \leq |S| - n$.*

Proof. Suppose that G is n -factor-critical. It is immediate that $|V(G)| \equiv n \pmod{2}$. Suppose there exists a vertex-set $S \subseteq V(G)$ with $|S| \geq n$, $o(G - S) > |S| - n$. Delete n vertices $v_1, v_2, \dots, v_n$ from S and denote the remaining set by S' . Then $o(G - \{v_1, v_2, \dots, v_n\} - S') = o(G - S) > |S| - n = |S'|$ and by Tutte's 1-Factor Theorem, $G - \{v_1, v_2, \dots, v_n\}$ has no 1-factors, a contradiction to the hypothesis.

Conversely, suppose that $|V(G)| \equiv n \pmod{2}$ and for any vertex-set $S \subseteq V(G)$ with $|S| \geq n$, $o(G - S) \leq |S| - n$. If G is not n -factor-critical, then

there exist n vertices $v_1, v_2, \dots, v_n$ such that $G - \{v_1, v_2, \dots, v_n\}$ has no 1-factors. Using Tutte's 1-Factor Theorem again, there exists a set $S' \subseteq V(G) - \{v_1, v_2, \dots, v_n\}$ so that $o(G - \{v_1, v_2, \dots, v_n\} - S') > |S'|$. Let $S = S' \cup \{v_1, v_2, \dots, v_n\}$. Then $o(G - S) > |S'| = |S| - n$, a contradiction. $\square$

Other characterizations for n -factor-critical graphs using M -alternating paths were obtained by Aldred *et al.* (2004) for the case of even n and Wen (2008) for the case of odd n , respectively.

Theorem 6.1.5 (Aldred *et al.* [17]). *Let G be a graph with a 1-factor M . Then G is $2r$ -factor-critical if and only if for any $2r$ vertices of G , there exist r disjoint M -alternating paths such that each of such paths joins two vertices of the $2r$ vertices, and starts and terminates with the edges in M .*

Theorem 6.1.6 (Wen [581]). *Let G be a graph of odd order with a near-perfect matching M and the unsaturated vertex x . Then G is $(2r + 1)$ -factor-critical if and only if the following conditions hold:*

- (a) *for any $2r$ vertices of $G - x$, $x_1, x_2, \dots, x_{2r}$, there exist r disjoint M -alternating paths, $P_1, P_2, \dots, P_r$ joining the vertices $x_1, x_2, \dots, x_{2r}$ in pairs and starting and terminating with the edges in M ;*
- (b) *for any $2r + 2$ vertices, $u_1 (= x), u_2, \dots, u_{2r+2}$, there exist $r + 1$ disjoint M -alternating paths, $P_1, P_2, \dots, P_{r+1}$, joining the vertices $u_1, u_2, \dots, u_{2r+2}$ in pairs such that P_1 starts at x and terminates at a vertex of $\{u_2, \dots, u_{2r+2}\}$ and starts with an edge in $E(G) - M$ and terminates with an edge in M ; P_i ($2 \leq i \leq r + 1$) starts and terminates with the edges in M .*

For $n = 1$, i.e., factor-critical graphs, Lovász [400] gave a structural description. He showed that a graph is factor-critical if and only if it has an odd ear decomposition (see Section 5.2). However, there are no ear decompositions or other similar decompositions existed for n -factor-critical graphs in general. It remains an interesting topic to investigate.

For bipartite graphs, there are many nice characterizations for k -extendable graphs using various concepts. For instance, Hall-type condition was observed by Plummer (1986).

Theorem 6.1.7 (Plummer [479]). *Let $G = (U, W)$ be a connected bipartite graph. Then the following are equivalent:*

- (a) *G is k -extendable;*
- (b) *$|U| = |W|$ and for all non-empty subsets X of U with $|X| \leq |U| - k$, $|N(X)| \geq |X| + k$;*
- (c) *For all $u_1, u_2, \dots, u_k \in U$ and $w_1, w_2, \dots, w_k \in W$, $G' = G - \{u_1, \dots, u_k, w_1, \dots, w_k\}$ has a 1-factor.*

Proof. (a) $\Rightarrow$ (b). Use induction on k . When $k = 1$, it is precisely Theorem 5.1.1. So assume $k \geq 2$. Let $\emptyset \neq X \subset U$ with $|X| \leq |U| - k$, but $|N(X)| \leq$

$|X| + (k-1)$. From Theorem 6.1.11, G is $(k-1)$ -extendable and thus $|N(X)| \geq |X| + (k-1)$ by induction hypothesis. Therefore, we have $|N(X)| = |X| + (k-1)$. Since $X \neq \emptyset$, it follows $|N(X)| = |X| + (k-1) \geq k$. Moreover, $|X| \leq |U| - k$ implies that $|U - X| = |U| - |X| \geq k$.

Since $|N(X)| \geq k$ and $|U - X| \geq k$, from Theorem 6.1.11 and Lemma 6.1.1, G is k -connected and thus there exist at least k disjoint paths in G connecting $N(X)$ and $U - X$. Since there are no edges between X and $W - N(X)$, these paths contain a matching of size at least k between $N(X)$ and $U - X$. Let M be a k -matching between $U - X$ and $N(X)$. Since G is k -extendable, M can be extended to a 1-factor of G which must match $|X| + k$ vertices into $N(X)$. Hence $|N(X)| \geq |X| + k$, a contradiction.

(b) $\Rightarrow$ (c). Suppose $|U| = |W|$ and let G' be defined as in the theorem. Let $U' = U - \{u_1, \dots, u_k\}$ and $W' = W - \{w_1, \dots, w_k\}$. Choose any $X' \subseteq U'$ with $X' \neq \emptyset$. Then $|X'| \leq |U'| = |U| - k$, and so $|N_{G'}(X')| \geq |X'| + k$ and thus $|N_{G'}(X')| \geq |X'|$. Hall's Theorem implies that G' has a 1-factor.

(c) $\Rightarrow$ (a). It is immediate from the definition. $\square$

The k -extendability of bipartite graphs can also be characterized by strongly connected digraphs. Recall that a digraph D is *strongly connected* if there exists a path from x to y and a path from y to x for any $x, y \in V(D)$, $x \neq y$. A set $S \subseteq V(D)$ is a *separator* if $D - S$ is not strongly connected and D is *k -strongly connected*, if $|V(D)| \geq k+1$ and D has no separators of order less than k .

Now we use $D(G, M)$ (see Section 5.1 for the details) to give a criterion of k -extendability. The proof present here is from [629].

Theorem 6.1.8 (Thomas [544]). *Let G be a bipartite graph and M a perfect matching of G . Then $D(G, M)$ is k -strongly connected if and only if G is k -extendable.*

Proof. Let D be k -strongly connected. We use induction on k to prove that G is k -extendable. When $k = 1$, it is not hard to verify it. Suppose that the conclusion holds for all integers $1 \leq m < k$. We prove that any k -matching M_0 of G is contained in a perfect matching of G .

Firstly we assume that $|M_0 \cap M| \geq 1$. Let $e \in M_0 \cap M$ and the vertex in D corresponding to e be v_e . Let $G' = G - V(e)$, $D' = D - v_e$, and $M' = M - e$. Then D' is $(k-1)$ -strongly connected and $D' = D(G', M')$. By the induction hypothesis, G' is $(k-1)$ -extendable. Hence $M_0 - \{e\}$, which is a $(k-1)$ -matching of G' , is contained in a perfect matching M' of G' . Then $M' \cup \{e\}$ is a perfect matching of G containing M_0 .

Next we deal with the case that $M_0 \cap M = \emptyset$. In this case, every edge in M_0 corresponds to an arc in D . Let the set of these arcs be A_0 , where $|A_0| = k$. Since D is k -strongly connected, for any k independent arcs there exists a set C_0 of mutually disjoint cycles in D containing all of them. C_0

corresponds to a set C of mutually disjoint M -alternating cycles in G . Then $C \triangle M$ is a perfect matching of G containing M_0 . Hence G is k -extendable.

Conversely, suppose that G is k -extendable. To see that D is k -strongly connected, let $\{v_1, v_2, \dots, v_{k-1}\}$ be a set of $k-1$ vertices in D . Denote by e_i the edge in G corresponds to v_i , $1 \leq i \leq k-1$. Let $G' = G - \bigcup_{i=1}^{k-1} V(e_i)$, $D' = D - \{v_i \mid 1 \leq i \leq k-1\}$ and $M' = m - \{e_i \mid 1 \leq i \leq k-1\}$. Then $D' = D(G', M')$. Since G is k -extendable, G' is 1-extendable. Hence D' is strongly connected and thus D is k -strongly connected. $\square$

In bipartite graphs, there are many properties shared by both of k -extendability and k -strongly connectivity, just as demonstrated in the last theorem, even though neither concept implies other. Below, we list two results from both camps as a comparison.

Theorem 6.1.9 (Menger (1927)). *Let D be a digraph. Then D is k -strongly connected if and only if $|V(D)| \geq k+1$ and D contains k internally vertex disjoint (s, t) -paths for every choice of distinct vertices $s, t \in V$.*

Theorem 6.1.10 (Aldred *et al.* [16]). *Let $G = (U, W)$ be a bipartite graph with a perfect matching. Then G is k -extendable if and only if for any perfect matching M and for each pair of vertices $u \in U$ and $w \in W$, there exist k internally disjoint M -alternating paths connecting u and w . Furthermore, these k paths start and end with edges in $E(G) - M$.*

Next we discuss the relations between k -extendable graphs and connectivity in arbitrary graphs. We often use the following version of Menger's Connectivity Theorem in the proofs of our results and thus state it as a lemma.

Lemma 6.1.1. *A graph G of order $p \geq 2n$ is n -connected if and only if for every two disjoint vertex-sets V_1 and V_2 of n vertices each, there exist n disjoint paths connecting V_1 and V_2 .*

The first two basic properties of k -extendable graphs were given by Plummer [478]. They served as the fundamental stones for matching extension theory.

Theorem 6.1.11 (Plummer [478]). *Let G be a graph of order $p \geq 2k+2$ and $k \geq 1$. If G is k -extendable, then*

- (a) G is $(k-1)$ -extendable; and
- (b) G is $(k+1)$ -connected.

Proof. (a) Assume that G is not $(k-1)$ -extendable. Let A be a set of $k-1$ independent edges which do not extend to a 1-factor. For any 1-factor M , the symmetric difference $M \triangle A$ consists of even cycles and at least two alternating paths, each of which has both end-vertices in M . Let P be the edge

set of one path. Then $P\Delta A$ is a set of k independent edges which can be extended to a 1-factor. Furthermore, this 1-factor contains at least one edge e not in $P\Delta A$ since $|P\Delta A| = k$ and $p \geq 2k + 2$. But then $A \cup \{e\}$ is a set of k independent edges which extends to a 1-factor containing A , a contradiction.

(b) We proceed with induction on k .

Consider the case of $k = 1$ first. Suppose that G is a 1-extendable graph but with a cutvertex u . Let the components of $G - u$ be $C_1, C_2, \dots, C_k$ ($k \geq 2$). Let e be any edge joining u to a vertex of C_1 . Then e can be extended to a 1-factor of G and thus $|V(C_1)|$ is odd and $|V(C_2)|, \dots, |V(C_k)|$ are all even. But then if f is an edge joining u to C_2 , it concludes that $|V(C_2)|$ is odd, a contradiction.

Now suppose that the result is true for $k - 1$, i.e., $(k - 1)$ -extendable graph is a k -connected. Let G be a k -extendable graph with $p \geq 2k + 2$. From (a), G must be $(k - 1)$ -extendable graph and thus k -connected by induction hypothesis.

Suppose that G is not $(k + 1)$ -connected. Then G has a cutset S of order k . Let $C_1, C_2, \dots, C_m$ be the components of $G - S$. Note first that $|V(C_1)| + |V(C_2)| + \dots + |V(C_m)| = |V(G)| - |S| \geq 2k + 2 - k = k + 2 \geq k + 1$, by Lemma 6.1.1 there exist k vertex-disjoint paths in G joining S to the vertices in $\cup_{i=1}^m V(C_i)$ and thus there are k independent edges, M , joining S to $\cup_{i=1}^m V(C_i)$.

First, suppose some C_i has $|V(C_i)| \geq k$. Then there exists a k -matching M from S and $V(C_i)$. Let $e = s_1 c_1 \in M$ with $s_1 \in S$. Since $S - \{s_1\}$ is not a cutset of G , there is an edge f joining s_1 to a vertex of a different component C_j ($j \neq i$). Let $M' = M - \{e\} \cup \{f\}$. Then M' is a k -matching in G . Because G is k -extendable, there is a 1-factor containing M and thus $|V(C_i)| - k$ is even. On the other side, there is a 1-factor containing M' and thus $|V(C_i)| - k + 1$ is even. This is a contradiction.

Now we consider a component C_j with $|V(C_j)| \leq k - 1$. Since $|V(G)| \geq 2k + 2$, by Lemma 6.1.1, there exists a k -matching M from S to $V(G) - S$ so that every vertex in $V(C_j)$ except one, say x , is saturated. That is, x is an isolated vertex in $G - V(M)$ or G is not k -extendable, a contradiction.

The induction is complete. $\square$

For n -factor-critical graphs, Liu and Yu [359], Favaron [202] independently obtained the following properties which are analogues of Theorem 6.1.11.

Theorem 6.1.12 (Liu and Yu [359]; Favaron [202]). *Let G be an n -factor-critical graph. Then*

- (a) G is $(n - 2)$ -factor-critical;
- (b) G is n -connected;
- (c) G is $(n + 1)$ -edge-connected. In particular, $\delta(G) \geq n + 1$.

Recall that a brick is a 3-connected bicritical graph and a brace is a 2-extendable bipartite graph. The next result provides a link between the 2-extendability and bricks/braces. Due to the importance of bricks/braces in the study of graph decompositions and the matching polytope/lattices (refer to Sections 5.3 and 5.4 for the details), this motivates the investigation of higher extendability, in particular, 2-extendability.

Theorem 6.1.13 (Plummer [478]). *If G is 2-extendable, then G is either a brick or a brace.*

Proof. Since a 2-extendable graph is 3-connected, to show the theorem we need only to show that G is either bicritical or bipartite.

Suppose, to the contrary, that G is not bicritical. By Theorem 6.1.4, there exists a set $S_0 \subseteq V(G)$ such that $o(G - S_0) \geq |S_0| - 1$ or $o(G - S_0) \geq |S_0|$ by the parity. On the other hand, since G has a 1-factor, by Tutte's 1-Factor Theorem, $o(G - S) \leq |S|$ for any subset S . Hence we have $o(G - S_0) = |S_0|$ or S_0 is a barrier. Therefore, $G - S_0$ has no even components and each 1-factor of G must match two vertices of S_0 with two vertices in different odd components of $G - S_0$. Since G is 1-extendable or every edge is in a 1-factor, S_0 is an independent set in G .

Let $O_1, O_2, \dots, O_r$ be all (odd) components of $G - S_0$, where $r = |S_0|$. If $|O_i| \geq 2$ for some i , since G is 3-connected by Theorem 6.1.11, there must be two independent edges e and g joining O_i to S_0 . Since G is 2-extendable, e and g can be extended to a 1-factor M of G . But, this case, two vertices of S_0 match to two vertices in the same component O_i , a contradiction. Therefore, $|O_1| = \dots = |O_r| = 1$. Recall that S_0 is independent and $G - S_0$ has no even components, thus G is a bipartite graph.

The above arguments imply that G is either bicritical or bipartite. $\square$

The k -extendable graphs form a rich family for studying and possess many nice properties. For instance, Plummer [478] showed that every graph with a relatively large degree is k -extendable. On the other hand, given girth g and extendability k , Gvozdjak and Nešetřil [232] present a construction of k -extendable graphs with girth at least g .

Next we discuss the minimum degree in a k -extendable graph. There is a forbidden range for which the minimum degrees can not be realized in k -extendable graphs. We need the following lemma from Ananchuen and Caccetta [38] first.

Lemma 6.1.2 ([38]). *Let G be a k -extendable graph on $2n$ vertices with $\delta(G) = k + t$, $1 \leq t \leq k \leq n - 1$. If $d_G(u) = \delta(G)$, then the subgraph $G[N_G(u)]$ has at most $t - 1$ independent edges.*

Theorem 6.1.14 (Ananchuen and Caccetta [44]). *If G is a k -extendable graph on $2n$ vertices ($1 \leq k \leq n - 1$), then either $k + 1 \leq \delta(G) \leq n$ or $\delta(G) \geq 2k + 1$.*

Proof. Suppose, to the contrary, that there exists a k -extendable graph G with $n + 1 \leq \delta(G) \leq 2k$. Let $d_G(u) = r = \delta(G)$. From Lemma 6.1.2, the subgraph $H = G[N_G(u)]$ contains a maximum matching M with $|M| \leq r - k - 1 \leq k - 1$. Furthermore, $|M| \geq 1$ as otherwise vertices of H have degree at most $2n - r < r$ in G .

Let F be a 1-factor in G containing M . Consider $M' = \{vw \in F \mid v, w \notin N_G(u)\}$. Clearly

$$\begin{aligned} |M'| &= \frac{1}{2}((2n - r - 1) - (r - 2|M| - 1)) \\ &= n - r + |M| \leq |M| - 1 \\ &\leq k - 2. \end{aligned}$$

If $G[\bar{N}_G(u) - V(M')]$ contains an edge e , then $M' \cup \{e\}$ is a matching of size at most $k - 1$ which does not extend to a 1-factor in G , since $H - V(M)$ is an independent set of order $r - 2|M|$ and

$$\begin{aligned} |\bar{N}_G(u) - V(M' \cup \{e\})| &= 2n - r - 1 - 2|M'| - 2 \\ &= 2n - r - 2(n - r + |M|) - 3 \\ &= r - 2|M| - 3. \end{aligned}$$

Hence, $G[\bar{N}_G(u) - V(M')]$ has no edges.

Now, since $|M| \leq r - k - 1$,

$$|\bar{N}_G(u) - V(M')| = r - 2|M| - 1 \geq 2k - r + 1 \geq 1.$$

Thus, there exists a vertex $x \in \bar{N}_G(u) - V(M')$. If $xy \notin E(G)$ for all $y \in V(M)$, then

$$\begin{aligned} d_G(x) &\leq 2|M'| + r - 2|M| \\ &\leq 2(n - r) + r = 2n - r \\ &< r, \end{aligned}$$

a contradiction. So $xy \in E(G)$ for some $y \in V(M)$. Let $yy' \in M$ and consider $M'' = M' \cup \{xy, y'u\}$. M'' is a matching of size at most k that does not extend to a 1-factor in G , since

$$|N_G(u) - \{y, y'\}| - |\bar{N}_G(u) - (V(M') \cup \{x\})| = (r - 2) - (r - 2|M| - 2) = 2|M|$$

and $H - \{y, y'\}$ has at most $|M| - 1$ independent edges. This contradicts the extendability of G and thus completes the proof of the theorem. $\square$

Can the values in the intervals be realized? Ananchuen (1992) constructed families of k -extendable graphs of order $2n$ with minimum degree j , for every $j \in [k + 1, n] \cup [2k + 1, 2n - 1]$.

When $n \geq 2k$, every value of $[k + 1, 2n - 1]$ can be realized. From Theorem 6.1.14, there does not exist a k -extendable graph of order $2n$ with minimum

degree j , for $n + 1 \leq j \leq 2k$. However, when the extendability is relatively large, the bound for minimum degrees can be sharpen quite a bit.

Theorem 6.1.15 (Lou and Yu [394]). *If G is a k -extendable graph on n vertices with $k \geq n/4$, then either G is bipartite or $\kappa(G) \geq 2k$.*

Proof. By contradiction. Suppose that G is a connected k -extendable graph with connectivity at most $2k - 1$ but not bipartite. Let S be a minimum cutset of G and M a maximum matching in $G[S]$. Let $T = S - V(M)$ and $r = |M|$. Since $|S| \leq 2k - 1$ and $|M| \leq k - 1$, by Observation 6.1.1 and Theorem 6.1.11 (b), $G - V(M)$ is $(k - r + 1)$ -connected. Then we have

$$|T| \geq k - r + 1 \geq 2 \quad (6.1)$$

and we have $2k - 1 \geq 2r + |T| \geq k + r + 1$, so

$$r \leq k - 2. \quad (6.2)$$

Claim 1. For every 1-factor F containing M , $F \cap E(G - S)$ is a maximum matching in $G - S$ and $|F \cap E(G - S)| \leq k - 1$.

In fact, since T is an independent set of G , by (6.1) and the assumption $|V(G)| \leq 4k$,

$$\begin{aligned} |F \cap E(G - S)| &= (|V(G)| - 2|M| - 2|T|)/2 \\ &= |V(G)|/2 - r - |T| \leq 2k - (k + 1) = k - 1. \end{aligned}$$

If $F \cap E(G - S)$ is not a maximum matching in $G - S$, then there is a matching F_1 in $G - S$ such that $|F_1| = |F \cap E(G - S)| + 1 \leq k$. It is not hard to see that the size of a maximum matching in $G - V(F_1)$ is at most $|V(G - S - V(F_1))|/2 + |M| \leq |V(G)|/2 - |F_1| - 1$, hence $G - V(F_1)$ does not have 1-factors, which contradicts the k -extendability of G .

By Claim 1 and the fact that T is an independent set of G , we can easily prove the following claim.

Claim 2. The size of every maximum matching in $G - S$ is $|V(G)|/2 - |M| - |T|$.

By (6.1), there are two distinct vertices x and y in T . By Theorem 6.1.12, the graph $H = G \cup xy$ is $(k - 1)$ -extendable. From (6.2), $M_1 = M \cup \{xy\}$ is a matching in H which has size at most $k - 1$. Then $H - V(M_1)$ has a 1-factor M^* and M^* matches each vertex of $T - \{x, y\}$ to a vertex in $V(G - S)$. Hence $M^* \cap E(G - S)$ is a matching of size $|V(G)|/2 - |M| - |T| + 1$ in $G - S$, a contradiction to Claim 2. The proof is complete. $\square$

The above theorem has many applications in the study of maximal/minimal k -extendable graphs which will be discussed in Section 7.1.

In the study of structure of k -extendable graphs, we often break down a k -extendable graph along its minimum cut-set and then investigate the properties of its components. The next theorem obtained by Maschlanka and Volkmann (1996) is a good tool for this purpose.

Theorem 6.1.16 (Maschlanka and Volkmann [421]). *Let k, t be positive integers and let G be a k -extendable graph with $\kappa(G) = k + t$. Then every vertex cut-set $S \subset V(G)$ with $|S| = k + t$ satisfies $\alpha(G[S]) \geq k + 2 - t$ or $G[S] = K_{2k}$. In particular, $\alpha(G) \geq 2k + 2 - \kappa(G)$.*

Finally, we discuss a concept related to 1-extendability. Let G be a graph with 1-factors and v be a vertex of G . Call v *totally covered* if every edge incident to v is allowed. Clearly, every vertex in a 1-extendable graph is totally covered. But how does the vertex-connectivity relate to the number of totally covered vertices? Zaks conjectured that every n -connected graph ($n \geq 2$) with a 1-factor has at least n totally covered vertices. Zaks confirmed the case of $n = 2$ and the conjecture was proved by Lovász (1972) and Mader (1973), independently.

Theorem 6.1.17 (Lovász [401]; Mader [419]). *If G is a k -connected graph with a 1-factor and $k \geq 2$, then G has at least k totally covered vertices.*

Proof. Without loss of generality, assume G is Φ -maximal. If not, we saturate G . Note that the set of allowed edges remains the same during the saturation process. Hence the Φ -maximal graph arising from G has no new totally covered vertices.

Let G_0 be the graph defined in Theorem 1.6.12 and suppose $\{x\}$ is a singleton class in $\mathcal{P}(G_0)$. Then tower $G_{\{x\}}$ must be empty or else x would be a cutvertex of G . But now every edge incident with x in G_0 is allowed in G_0 , since $\{x\}$ is a class of $\mathcal{P}(G)$, and hence is also allowed in G ; that is, x is totally covered in G .

Now if every class of $\mathcal{P}(G_0)$ is a singleton, then $G = G_0$. Since a k -connected graph must have at least $k + 1$ vertices, the theorem follows immediately. So suppose S is a class of $\mathcal{P}(G_0)$ and that S has at least two vertices. Then, by Theorem 5.1.6 (c), S is a cutset of G_0 , and hence of G as well by Theorem 1.6.12. Thus $|S| \geq k$ by the hypothesis and there must be at least $|S|$ one-element classes in $\mathcal{P}(G_0)$ and the proof is complete. $\square$

Gabow [221] designed a polynomial algorithm to find all totally covered vertices by using depth-first version of a cardinality matching algorithm. Consequently, he gave another proof to Theorem 6.1.17. Moreover, Mader (1973) actually proved a more general version of Zaks's conjecture.

Theorem 6.1.18 (Mader [419]). *If G is a 2-connected graph with a 1-factor, then G has at least $\delta(G)$ totally covered vertices.*

From Theorem 6.1.17, we can have the following two interesting consequences.

Corollary 6.1.1 (Lovász [401]). *If G is a k -connected graph and $\Phi(G \cup e) > \Phi(G)$ for all edges $e \notin E(G)$, then G contains at least k vertices v such that $G - v$ is factor-critical.*

Corollary 6.1.2 (Gabow [221]). *Let G is a k -connected graph with a 1-factor. If it has exactly k totally covered vertices, then these vertices form an independent set.*

6.2 Equivalence and recursive relationships

Different types of matching extensions are closely related. It is interesting to notice the fact that if a graph G is $2k$ -factor-critical then it must be k -extendable. Furthermore, if a graph G is $2k$ -factor-critical, then it is still k -extendable after adding any number of edges to it. In this sense, $2k$ -factor-criticality is a much stronger concept than k -extendability.

Many results about k -extendability can be extended to $2k$ -factor-criticality under the same conditions. On the other hand, not all k -extendable graphs are $2k$ -factor-critical. For instance, the complete bipartite graph $K_{k+1, k+1}$ is k -extendable but is not even 2-factor-critical. However, for non-bipartite graphs we have the following result.

Theorem 6.2.1 (Yu [617]). *Let G be a connected non-bipartite graph of even order n .*

- (a) *If G is $2m$ -extendable, then it is also $2m$ -factor-critical;*
- (b) *if G is m -extendable and $m \geq \frac{1}{3}(n+1)$, then G is $2m$ -factor-critical.*

Proof. (a) Let $S = \{x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_m\}$ be a set of $2m$ vertices of G . Adding the edges $x_i y_i$ ($1 \leq i \leq m$) to G if $x_i y_i$ is not an edge of G and we denote the resulting graph by G' . Then $|E(G')| - |E(G)| \leq m$ and G' is again non-bipartite. By Theorem 7.1.2, then G' is m -extendable. Therefore there exists a 1-factor containing edges $x_1 y_1, x_2 y_2, \dots, x_m y_m$ in G' or $G - S$ has a 1-factor. That is, G is $2m$ -factor-critical.

(b) For any vertex set $S \subseteq V(G)$ with $|S| = 2m$, we need to show that $G - S$ has a 1-factor. We show a claim first.

Claim. G is $(n - 2m)$ -factor-critical.

Case 1. $2m \equiv 0 \pmod{4}$

Since G is m -extendable, from Part (a), G is m -factor-critical. By the assumption $m \geq \frac{1}{3}(n+1) \geq \frac{1}{3}n$, we have $3m \geq n$ or $m \geq n - 2m$. As G is m -factor-critical, by Theorem 6.1.12, so G is $(n - 2m)$ -factor-critical.

Case 2. $2m \equiv 2 \pmod{4}$

Since G is m -extendable, G is $(m-1)$ -extendable. By Part (a), G is $(m-1)$ -factor-critical as $m-1$ is even. By the assumption $m \geq \frac{1}{3}(n+1)$ or $3m \geq n+1$, we have $m-1 \geq n-2m$. As G is $(m-1)$ -factor-critical, by Theorem 6.1.12, then G is $(n-2m)$ -factor-critical (note that $m-1$ and $n-2m$ have the same parity).

Since G is $(n-2m)$ -factor-critical and $|G-S| = n-2m$, so there exists a 1-factor in $G - (G-S) = S$, say M . But G is m -extendable and M is an m -matching in S , so M can be extended to a 1-factor of G , that is, there is a 1-factor in $G-S$. $\square$

Recently, Zhang, Wang and Lou [631] improved Theorem 6.2.1 (b) and showed that if $k \geq (|V(G)|+2)/4$, then a graph G is k -extendable if and only if it is $2k$ -factor-critical. Furthermore, the equivalence of $k\frac{1}{2}$ -extendability and $(2k+1)$ -factor-criticality was obtained.

There is a characterization of $2k$ -factor-criticality by Lou (2000).

Theorem 6.2.2 (Lou [387]). *A graph G of even order is $2k$ -factor-critical if and only if*

- (a) G is k -extendable; and
- (b) for any edge set $D \subseteq E(\bar{G})$, $G \cup D$ is k -extendable.

Proof. $(\Rightarrow)$ Suppose G is $2k$ -factor-critical. Clearly, G is also k -extendable. Let D be an edge-set such that $D \subseteq E(\bar{G})$ and let $G' = G \cup D$. Since adding new edges to G the number of odd components in $G-S$ does not increase for any set $S \subseteq V(G)$, we have $o(G'-S) \leq |S| - 2k$ for any $S \subseteq V(G')$ with $|S| \geq 2k$. By Theorem 6.1.2, G' is k -extendable.

$(\Leftarrow)$ Suppose G is not $2k$ -factor-critical. From Theorem 6.1.4, there exists a set $S \subseteq V(G)$ with $|S| \geq 2k$ such that $o(G-S) > |S| - 2k$ or $o(G-S) \geq |S| - 2k + 2$ by parity. Consider two cases.

Case 1. $G[S]$ contains k independent edges.

Then by Theorem 6.1.2, G is not k -extendable. So (a) of this theorem does not hold, a contradiction.

Case 2. $G[S]$ contains less than k independent edges.

However, $|S| \geq 2k$. We can add an edge-set $D \subseteq E(\bar{G})$ so that $G[S]$ contains k independent edges. Let $G' = G \cup D$. Since $o(G-S) \geq |S| - 2k + 2$, clearly $o(G'-S) \geq |S| - 2k + 2$. By Theorem 6.1.2, G' is not k -extendable, a contradiction to (b) of the theorem. $\square$

Naturally, we would like to identify all k -extendable graphs. However, from the known families of k -extendable graphs, it may be a challenge to list all of them. But Ananchuan and Caccetta [43] managed to characterize $(n-1)$ -

and $(n - 2)$ -extendable graphs of order $2n$. For instance, a graph is $(n - 1)$ -extendable if and only if it is $K_{n,n}$ or K_{2n} .

It seems that there are “genes” built in k -extendable and n -factor-critical graphs. In other words, if smaller subgraphs have one property, then the original graph also carries the same property. Nishimura and Saito [459] showed that if $G - V(e)$ is k -extendable for each $e \in E(G)$, then G itself is k -extendable. For factor-criticality, Favaron and Shi [206] proved that if $G - V(e)$ is n -factor-critical for each $e \in E(G)$, then G is n -factor-critical. Nishimura [461] improved both results and showed that it is enough to check just a few subgraphs instead of all subgraphs $G - V(e)$.

Theorem 6.2.3 (Nishimura [461]). *Let G be a graph.*

- (a) *Let F be an arbitrary (but fixed) 1-factor. If $G - V(e)$ is k -extendable for each edge $e \in F$, then G is k -extendable.*
- (b) *Let M be an arbitrary (but fixed) maximal matching of G . If $G - V(e)$ is n -factor-critical for each edge $e \in M$, then G is n -factor-critical.*

To show this result, we need a lemma first which has its own merit and interest. Since the proof ideas for both (a) and (b) are quite similar (but the proof of part (a) requires more detailed treatments), so we only present the proof of (b) here.

Lemma 6.2.1. *Let G be a graph with at least $k + 4$ vertices and let e and f be two independent edges of G . If $G - V(e)$ and $G - V(f)$ are k -connected, then G is k -connected.*

The proof of Theorem 6.2.3 (b). Let G be a graph satisfying the conditions of the theorem. If $n = 0$, then clearly the theorem holds. Now we consider $n \geq 1$. Since $G - V(e)$ is n -factor-critical, by Theorem 6.1.12 (b), $G - V(e)$ is n -connected and thus $|V(G)| = |V(G) - V(e)| + 2 \geq n + 2 + 2 = n + 4$. From Lemma 6.2.1, G is connected.

Without loss of generality, assume that the size of any maximal matching is at least 2. Suppose that there exists a maximal matching M such that $G - V(e)$ is n -factor-critical for each edge $e \in M$ but G is not n -factor-critical. Then for some vertex-set R of order n , $G - R$ has no 1-factors. From Tutte’s 1-Factor Theorem, there exists a subset $S \subseteq V(G) - R$ so that $o(G - R - S) \geq |S| + 2$. Let $W = G - R - S$.

Claim 1. $M \subset E(R, G - R)$.

If there exists an edge $e \in M \cap E(R)$, then we have $o((G - V(e)) - (R - V(e)) - S) = o(W) \geq |S| + 2$, which implies that $G - V(e)$ is not $(n - 2)$ -factor-critical and therefore, by Theorem 6.1.12 (a), $G - V(e)$ is not n -factor-critical, a contradiction.

If there exists an edge $e \in M \cap (E(G[S]) \cup E(G[C_i]))$, where C_i is one of components of W , then $o(G - R - S - V(e)) \geq o(G - R - S) \geq |S| + 2$ and hence $G - V(e)$ is not $(n - 2)$ -factor-critical, a contradiction again.

If there exists an edge $e \in M \cap E(S, C_i)$, where C_i is one of components of W , then $o(G - R - S - V(e)) \geq o(G - R - S) - 1 \geq |S| + 1$ and hence $G - V(e)$ is not $(n - 2)$ -factor-critical, a contradiction.

Claim 2. $S = \emptyset$ and $W = G - R$ has no even components.

Suppose $S \neq \emptyset$. Since $M \neq \emptyset$, there exists an edge $e = ab \in M$ satisfying $e \in E(S, R)$ or $e \in E(W, R)$. Let $a \in S \cup W$ and $b \in R$. If $e \in E(S, R)$, then for a vertex $c \in W$, we have $o((G - V(e)) - (R \cup \{c\} - \{b\}) - (S - \{a\})) \geq o(W) - 1 \geq |S| + 1$, which implies $G - V(e)$ is not n -factor-critical, a contradiction. If $e \in E(W, R)$, then for a vertex $d \in S$, we have $o((G - V(e)) - (R \cup \{d\} - \{b\}) - (S - \{d\})) \geq o(W) - 1 \geq |S| + 1$. Again, we have a contradiction. Thus, we have $S = \emptyset$.

Suppose that W has an even component D . Since G is connected and M is maximal, there exists an edge $e = ab \in M \cap E(D, R)$, where $a \in D$ and $b \in R$. Then, since $|D| \geq 2$, for a vertex $c \in D - \{a\}$, $(G - V(e)) - (R \cup \{c\} - \{b\})$ has all odd components of $G - R$, i.e., we have $o((G - V(e)) - (R \cup \{c\} - \{b\})) \geq o(G - R) \geq 2$, a contradiction.

Claim 3. $o(G - R) = 2$.

Let the odd components of $G - R$ be $C_1, C_2, \dots, C_m$, where $m = o(G - R)$. Suppose that $m \geq 4$ since m is even. Without loss of generality, we assume $e = ab \in M \cap E(C_1, R)$. Let $a \in C_1$, $b \in R$ and $c \in V(C_2)$. Then $(G - V(e)) - (R \cup \{c\} - \{b\})$ has odd components $C_3, \dots, C_m$ of $G - R$, i.e., $o((G - V(e)) - (R \cup \{c\} - \{b\})) \geq m - 2 \geq 2$, which implies that $G - V(e)$ is not n -factor-critical, a contradiction.

Since $|V(G)| \geq n + 4$ and $|R| = n$, we have $|C_1 \cup C_2| \geq 4$. Furthermore, since C_1 and C_2 are odd components, $|C_1| \geq 3$ or $|C_2| \geq 3$. We may assume $|C_1| \geq 3$. By Claim 1, $M \subset E(R, G - R)$, the maximality of M , and the connectedness of G , there exists an edge $e \in M \cap E(C_1, R)$. Let $e = ab$, $a \in C_1$, $b \in R$. Then, we can take a vertex $c \in C_1 - \{a\}$ so that $o((G - V(e)) - (R \cup \{c\} - \{b\})) \geq o(G - R) = 2$. This shows $G - V(e)$ is not n -factor-critical, a contradiction.

We complete the proof. $\square$

We have discussed extendability/factor-criticality in a graph with that in its subgraphs. We continue to investigate these nice recursive relations in various subgraphs. We start with a result from Nishimura [456]. Its special case (i.e., $k = 0$) was obtained by Maschlanka and Volkmann [421] first.

Theorem 6.2.4 (Nishimura [456]). *Let G be a connected graph of order $2n$ ($n \geq 3$), and let r and k be integers such that $n > r > k \geq 1$. If for some integer r , every induced connected subgraph of order $2r$ is k -extendable, then G is k -extendable.*

Theorem 6.2.5 (Nishimura [456]). *Let G be a connected graph of order $2n$. Let r and k be positive integers such that $n - r \geq k + 1$. If $G - S$ is k -extendable for every connected subset S of order $2r$, then G is k -extendable.*

In the discussion of extendability of a graph and its subgraphs, there are several variations on what kind of induced subgraphs to be used or what properties to acquire. The following are some of results in this nature.

Theorem 6.2.6 (Nishimura and Saito [459]). *If a graph G has the property that $G[S]$ is k -extendable for every connected subset S of order $2r$, then $G[T]$ is $(k+1)$ -extendable for every connected subset T of order $2(r+1)$.*

A connected graph G is called $\langle r, k \rangle$ -extendable if $G - S$ is k -extendable for every connected subset S of order $2r$ for which $G - S$ is connected.

Theorem 6.2.7 (Nishimura [460]). *Let n, r and k be positive integers with $n - r > k$. Then every 2-connected $\langle r, k \rangle$ -extendable graph of order $2n$ is $\langle r - 1, k \rangle$ -extendable.*

A connected graph G is called $\langle r : m, k \rangle$ -extendable if for every connected subset S of order $2r$ for which $G - S$ is connected, $G[S]$ is m -extendable and $G - S$ is k -extendable.

Theorem 6.2.8 (Chan and Nishimura [128]). *Let n, r, m and k be positive integers with $n - r > k$ and $r > m$. Then every 2-connected $\langle r : m, k \rangle$ -extendable graph of order $2n$ is $\langle r + 1 : m + 1, k - 1 \rangle$ -extendable.*

6.3 Matching extension and graphic parameters

In this section, we focus mainly on two topics, the existence of k -extendable or n -factor-critical graphs in terms of various graphic parameters, and the existence in special families of graphs.

6.3.1 Matching extension and forbidden subgraphs

We discuss the existence of k -extendability under conditions involved forbidden subgraphs, in particular, $K_{1,n}$ -free graphs.

Plummer [488; 490] started the study of extendability in claw-free graphs, in particular, 2-extendability of 3-connected claw-free graphs and 4-regular 4-connected claw-free graphs. Here we only mention the following interesting result.

Theorem 6.3.1 (Plummer [490]). *Let G be a 3-connected claw-free graph of even order. Then G is 2-extendable if and only if G does not contain two independent edges $e_i = a_i b_i$ ($i = 1, 2$) such that $G - \{a_1, a_2, b_1, b_2\}$ consists of precisely two components which are both odd.*

With the high vertex-connectivity, we can obtain the following elegant result for claw-free graphs.

Theorem 6.3.2 (Plummer [488]). *Suppose $k \geq 0$ and G is a $(2k + 1)$ -connected claw-free graph of even order. Then G is $2k$ -factor-critical. In particular, G is k -extendable and when $k = 1$ G is a brick.*

Proof. Let S be any subset of $V(G)$ with order $2k$ and $G' = G - S$. Then G' is connected and claw-free since G is a $(2k + 1)$ -connected claw-free graph. From Theorem 1.4.4, the graph G' has a 1-factor, i.e., G is $2k$ -factor-critical. $\square$

Theorem 6.3.2 is sharp in the sense that there exists a family of infinite graphs which is claw-free, $2k$ -connected and has an even number of vertices, but is not k -extendable.

Similar to the above result, we can change the claw-free condition a little bit to obtain a sufficient condition for factor-critical graphs.

Theorem 6.3.3 (Lou and Yu [395]). *Let G be a graph of order p and n an integer such that $0 \leq n \leq p - 2$ and $p \equiv n \pmod{2}$. If G is $(2n + r)$ -connected and $K_{1, n+r+2}$ -free, then G is n -factor-critical where $2n + r \geq 1$.*

Proof. Suppose that G is not n -factor-critical. Since G is $(2n + r)$ -connected, by Theorem 6.1.4, there exists a subset $S \subseteq V(G)$ with $|S| \geq 2n + r$ such that

$$\omega(G - S) \geq o(G - S) \geq |S| - n + 2 \geq 2. \quad (6.3)$$

Let $C_1, C_2, \dots, C_t$ be the components of $G - S$, where $t = \omega(G - S)$. Since G is $K_{1, n+r+2}$ -free, each vertex u in S is adjacent to at most $n + r + 1$ components of $G - S$. Then we have $e_G(S, G - S) \leq |S|(n + r + 1)$. By the $(2n + r)$ -connectedness of G , each C_i is adjacent to at least $2n + r$ vertices in S . Then $e_G(S, G - S) \geq t(2n + r)$. Therefore, $t(2n + r) \leq |S|(n + r + 1)$. Recall $|S| \geq 2n + r$ and thus we have

$$\omega(G - S) = t \leq \frac{|S|(n + r + 1)}{2n + r} = |S| - \frac{n - 1}{2n + r}|S| \leq |S| - n + 1,$$

a contradiction to (6.3). $\square$

On the other hand, we have the following

Theorem 6.3.4 (Plummer [488]). *Suppose $k \geq 1$ and G is a claw-free k -extendable graph of even order. Then $\delta(G) \geq 2k$.*

Ryjáček [508] generalized the notion of “claw-free” to “almost claw-free”. A graph G is *almost claw-free* (ACF for short) if the neighborhood of every claw center has a dominating set of size ≤ 2 and the set of all claw centers is independent. For two nonadjacent vertices a and b in G , let $J(a, b) = \{y \in N(a) \cap N(b) \mid N[y] \subseteq N[a] \cup N[b]\}$. The vertices of $J(a, b)$ are called

the *dominators* of the pair $\{a, b\}$. Note that the property of almost claw-free can be checked in polynomial time. He also proved that an almost claw-free graph G is $K_{1,5}$ -free and has a 1-factor when the order is even. Furthermore, he generalized Theorems 6.3.2 and 6.3.3 to the following.

Theorem 6.3.5 (Ryjáček [509]). *Let G be a $(2k + 1)$ -connected graph of even order.*

- (a) *If G is $K_{1,k+3}$ -free and the claw centers are independent, then G is k -extendable;*
- (b) *if G is almost claw-free, then G is k -extendable.*

A graph G is *quasi-claw-free* (QCF for short) if every pair of vertices $\{a, b\}$ at distance 2 satisfies the condition $J(a, b) \neq \emptyset$. In particular, every claw-free graph is quasi-claw-free.

A claw $\langle \{z; a_1, a_2, a_3\} \rangle$ is said to be *dominated* if $J(a_1, a_2) \cup J(a_2, a_3) \cup J(a_1, a_3) \neq \emptyset$. A graph G has *dominated claw toes*, or G is a *DCT*-graph for short, if every claw in G is dominated. Clearly, $QCF \subset DCT$ and also $ACF \subset DCT$.

The following theorem is proved in [204], we present a shorter proof here.

Theorem 6.3.6 (Favaron [204]). *Let G be a $(k + 1)$ -connected DCT-graph, and $|V| \equiv k \pmod{2}$. Then G is k -factor-critical.*

Proof. Suppose, to the contrary, that G is not k -factor-critical. Then there exists a minimal set S with $|S| \geq k$ such that

$$o(G - S) > |S| - k,$$

by parity, $o(G - S) \geq |S| - k + 2 \geq 2$. So S is a cutset of G . Since G is $(k + 1)$ -connected, then $|S| \geq k + 1$ and $o(G - S) \geq |S| - k + 2 \geq 3$. By the minimality of S , every vertex in S is adjacent to at least 3 odd components of $G - S$. Let $\langle \{z; a_1, a_2, a_3\} \rangle$ be a claw with claw center $z \in S$ and $a_i \in C_i$. Clearly, $J(a_1, a_2) = \emptyset$, $J(a_2, a_3) = \emptyset$, $J(a_1, a_3) = \emptyset$. Therefore, $J(a_1, a_2) \cup J(a_2, a_3) \cup J(a_1, a_3) = \emptyset$, a contradiction. Therefore, G is k -factor-critical. $\square$

Next we discuss the forbidden subgraph involved with other graphic parameter, dominating-criticality.

Recall that a set $S \subset V$ is a *dominating set* of G if every vertex in V is either in S or is adjacent to a vertex in S . We denote the dominating number by $\gamma(G)$. A graph G is *domination vertex critical*, if $\gamma(G - v) < \gamma(G)$ for every vertex v in G . Indeed, if $\gamma(G - v) < \gamma(G)$, then $\gamma(G - v) = \gamma(G) - 1$. A graph G is *domination edge critical*, if $\gamma(G \cup e) < \gamma(G)$ for any edge $e \notin E(G)$. We call a graph G is *k - γ -vertex-critical* (resp. *k - γ -edge-critical*) if it is domination vertex critical (resp. domination edge critical) and $\gamma(G) = k$.

The subject of γ -vertex-critical graphs was studied first by Brigham, Chinn and Dutton [92] and continued by Fulman *et al.* [? 220]. Clearly, the only 1-vertex-critical graph is K_1 (a single vertex). Brigham, Chinn and Dutton [92] pointed out that the 2-vertex-critical graphs are precisely the family of graphs obtained from the complete graphs K_{2n} with a 1-factor removed. For $\gamma > 2$, however, much remains unknown about the structure of γ -vertex-critical graphs.

The following result was conjectured by Wojcicka, and was proved by Favaron *et al.* ([207; 211; 549]). (In Mynhardt's survey paper, she referred it as "Wojcicka's Theorem".)

Theorem 6.3.7 (Wojcicka's Theorem [207; 211; 549]). *Every connected $3\text{-}\gamma$ -edge-critical graph with $\delta(G) \geq 2$ has a hamiltonian cycle.*

By Wojcicka's theorem, if $|V|$ is odd, then G is factor-critical; if $|V|$ is even, then it is an elementary graph.

Recently, Ananchuen and Plummer [47; 48; 51] began to study matchings in 3-vertex-critical graphs. They showed that a $K_{1,5}$ -free 3-vertex-critical graph of even order has a 1-factor (see [48]) and a $K_{1,4}$ -free 3-vertex-critical graph of odd order is factor-critical (see [51]).

Wang and Yu [577] proved the following result, which was conjectured in [51].

Theorem 6.3.8 (Wang and Yu [577]). *If G is a $K_{1,5}$ -free 3-vertex-critical 2-connected graph of odd order with $\delta(G) \geq 3$, except the graphs G_1 and G_2 shown in Fig. 6.2, then G is factor-critical.*

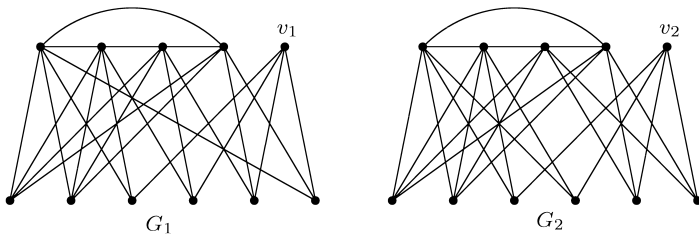


Fig. 6.2 Graphs G_1 and G_2 .

For k - γ -edge-critical graph, Wang and Yu [576] proved the following result which was conjectured in [52]. For the special cases of $k = 2$ and $k = 3$, it was confirmed by Ananchuen and Plummer in [47; 52].

Theorem 6.3.9 (Wang and Yu [576]). *Let G be a graph with $\delta(G) \geq k+1$ ($k \geq 2$) and $k \equiv |V(G)| \pmod{2}$. If G is a k -connected $3\text{-}\gamma$ -edge-critical claw-free graph, then G is k -factor-critical.*

6.3.2 Matching extension and toughness

Let $G \not\cong K_n$ be a graph and S a vertex cutset in G . Recall that the *toughness* of G is defined

$$t(G) = \min \frac{|S|}{\omega(G - S)},$$

where the minimum is taken over all cutsets S of G .

The next two results are samples of properties under this graphic parameter.

Theorem 6.3.10 (Plummer [481]). *Let G be a graph of even order p and k an integer with $1 \leq k < p/2$. If $t(G) > k$, then G is k -extendable.*

Proof. First suppose that $k = 1$. Note that since $t(G) > 1$, G has a 1-factor by Tutte's 1-Factor Theorem.

To see that G is 1-extendable, we show it by contradiction. Suppose that there exists an edge $e = xy \in E(G)$ not belonging to any 1-factor. Let $G' = G - x - y$. Then there is a set $S' \subseteq V(G')$ such that $|S'| < o(G' - S')$ or $|S'| \leq o(G' - S') - 2$ by parity.

Set $S_0 = S' \cup \{x, y\}$. Since G has a 1-factor, it follows that $o(G - S_0) \leq |S_0| = |S'| + 2$. On the other hand, $G - S_0 = G' - S'$ and $|S'| \leq o(G' - S') - 2$. Thus $o(G - S_0) = |S_0|$. But then

$$t(G) = \min_{S \subseteq V(G)} \frac{|S|}{\omega(G - S)} \leq \frac{|S_0|}{\omega(G - S_0)} \leq \frac{|S_0|}{o(G - S_0)} = 1$$

contradicting to $t(G) > 1$. So the desired conclusion holds when $k = 1$.

Now suppose $k \geq 2$, and assume that G is not k -extendable. Let $M = \{e_1, \dots, e_k\}$ be a k -matching which does not extend to a 1-factor. Denote $e_i = x_i y_i$ for $i = 1, \dots, k$ and $G_1 = G - \{x_1, \dots, x_k, y_1, \dots, y_k\}$. Hence G_1 has no 1-factors and, by Tutte's Theorem, there exists a set $S_1 \subseteq V(G_1)$ such that $|S_1| < o(G_1 - S_1)$ or $|S_1| \leq o(G_1 - S_1) - 2$ by parity.

Now $G_2 = G - \{x_k, y_k\}$ has a 1-factor since we have already proved that G is 1-extendable. Let $S_2 = S_1 \cup \{x_1, \dots, x_{k-1}, y_1, \dots, y_{k-1}\}$ and note that once again by Tutte's Theorem, $o(G_2 - S_2) \leq |S_2| = |S_1| + 2k - 2 \leq o(G_1 - S_1) + 2k - 4$. But $G_2 - S_2 = G_1 - S_1$ and so it follows that $o(G_2 - S_2) \geq |S_1| + 2$.

Now let $S_3 = S_1 \cup \{x_1, \dots, x_k, y_1, \dots, y_k\}$. Then, since $G - S_3 = G_2 - S_2$, we have

$$\begin{aligned} t(G) &= \min_{S \subseteq V(G)} \frac{|S|}{\omega(G - S)} \leq \frac{|S_3|}{\omega(G - S_3)} = \frac{|S_3|}{\omega(G_2 - S_2)} \\ &\leq \frac{|S_3|}{o(G_2 - S_2)} = \frac{|S_1| + 2k}{o(G_2 - S_2)} \\ &\leq \frac{|S_1| + 2k}{|S_1| + 2} \leq \frac{k|S_1| + 2k}{|S_1| + 2} = k, \end{aligned}$$

again a contradiction to the hypothesis and thus complete the proof. $\square$

The lower bound for toughness condition is sharp. Let H be a graph obtained by joining each vertex of K_{2k} to each of vertices of two disjoint copies of K_{2k+1} . Then $t(H) = k$, but H is not k -extendable.

Liu and Yu [360] observed that Plummer's proof for Theorem 6.3.10 can be easily modified to show that $t(G) > k$ implies $2k$ -factor-criticality.

It is hard to resist not mentioning the well-known result obtained by Enomoto *et al.* [186] that every k -tough graph G with $k|G|$ even has a k -factor. As pointed out by Plummer that this result and Theorem 6.3.10 are independent, i.e., neither implies other. On the other hand, Liu and Yu [360] showed the following.

Theorem 6.3.11 (Liu and Yu [360]). *Let G be a graph of even order p , where $p \geq 2k + 2$. If $t(G) \geq k \geq 1$, then G is $(2k - 1)$ -edge-deletable. That is, after deleting any $2k - 1$ edges of G , the resulting graph still has a 1-factor.*

Proof. If G is a complete graph, it is easy to check that G is $(p - 2)$ -edge-deletable and thus $(2k - 1)$ -edge-deletable since $p \geq 2k + 2$. Assume that G is not a complete graph. Let E' be any subset of $E(G)$ with $|E'| = 2k - 1$. Set $G' = G - E'$. From Tutte's Theorem, we need only to prove that for any proper subset $S \subseteq V(G)$, $o(G' - S) \leq |S|$ or

$$\frac{|S|}{o(G' - S)} \geq 1. \quad (6.4)$$

Since $t(G) \geq k$, the edge-connectivity of G is at least $2k$ (see Chvátal [151]). Hence G' is connected. We consider two cases.

Case 1. S is not a vertex cutset of G .

In this case, $G - S$ is connected. So $o(G - S) \leq \omega(G - S) = 1$. When $|S| \geq 2k$, we have

$$\frac{|S|}{o(G' - S)} \geq \frac{|S|}{\omega(G' - S)} \geq \frac{|S|}{\omega(G - S) + 2k - 1} \geq \frac{2k}{2k} = 1.$$

When $|S| = r \leq 2k - 1$, we have $\kappa'(G - S) \geq \kappa(G - S) \geq 2k - r$. Thus $\omega(G' - S) \leq \omega(G - S) + 2k - 1 - (2k - r) = 1 + r - 1 = r$. Therefore

$$\frac{|S|}{o(G' - S)} \geq \frac{|S|}{\omega(G' - S)} \geq \frac{r}{r} = 1.$$

Case 2. S is a vertex cutset of G .

In this case, we have $\frac{|S|}{\omega(G - S)} \geq t(G) \geq k$, that is, $|S| \geq k \omega(G - S)$. Thus

$$\frac{|S|}{o(G' - S)} \geq \frac{|S|}{\omega(G' - S)} \geq \frac{|S|}{\omega(G - S) + 2k - 1}. \quad (6.5)$$

Case 2.1. $|S| \geq k \omega(G - S) + 1$.

Since $k \omega(G - S) + 1 - (\omega(G - S) + 2k - 1) = (k - 1)\omega(G - S) - 2k + 2 \geq 2(k - 1) - 2k + 2 = 0$. By (6.5), we have

$$\frac{|S|}{o(G' - S)} \geq \frac{|S|}{\omega(G - S) + 2k - 1} \geq \frac{\omega(G - S)k + 1}{\omega(G - S) + 2k - 1} \geq 1.$$

Case 2.2. $|S| = k \omega(G - S)$.

If $\omega(G - S) = 2$, we have $|S| = 2k$ and $|V(G - S)|$ is even. Suppose that (6.4) does not hold. Then $o(G' - S) > |S|$ and by parity we have $o(G' - S) \geq |S| + 2 = 2k + 2$. Thus

$$2k + 2 \leq o(G' - S) \leq \omega(G' - S) \leq \omega(G - S) + 2k - 1 = 2 + 2k - 1 = 2k + 1,$$

which is impossible. Hence, in this case again we have $\frac{|S|}{o(G' - S)} \geq 1$.

If $\omega(G - S) \geq 3$ and $k \geq 2$, we have

$$|S| = k \omega(G - S) \geq \omega(G - S) + 2k - 1.$$

From (6.5),

$$\frac{|S|}{o(G' - S)} \geq \frac{|S|}{\omega(G - S) + 2k - 1} \geq 1.$$

Finally, we assume that $\omega(G - S) \geq 3$ and $k = 1$. Then $|S| = k \omega(G - S) = \omega(G - S)$. By a simple parity argument, we can see that $\omega(G - S) \equiv |S| \equiv o(G - S) \pmod{2}$ or $\omega(G - S) \equiv o(G - S) \pmod{2}$. That is, the number of even components of $G - S$ is even. By noticing $|E'| = 2k - 1 = 1$, we have $o(G' - S) \leq o(G - S) + 2$ and $\omega(G' - S) \leq \omega(G - S) + 1$. If $G - S$ has at least two even components, then

$$\frac{|S|}{o(G' - S)} \geq \frac{|S|}{o(G - S) + 2} \geq \frac{|S|}{\omega(G - S)} = 1.$$

If $G - S$ has no even components, then $o(G' - S) = o(G - S) = \omega(G - S)$. So $\frac{|S|}{o(G' - S)} = \frac{|S|}{\omega(G - S)} = 1$.

So we conclude that (6.4) holds in all cases, i.e., G' has a 1-factor and thus G is $(2k - 1)$ -edge-deletable. $\square$

Remark 6.3.1. There are many sufficient conditions for graphs being n -factor-critical in terms of degree, planarity, toughness, claw-free, etc. In fact, one might note that most known sufficient conditions for k -extendable graphs can be generalized to n -factor-critical graphs with minor modifications.

6.3.3 Matching extension in planar graphs and surfaces

In this section, we study extendability of the graphs embedded on surfaces. Plummer (1989) started with the planar graphs first and showed that this class of graphs do not have extendability more than two.

Theorem 6.3.12 (Plummer [483]). *No planar graph is 3-extendable.*

From the above theorem, we need only to investigate 1- or 2-extendable planar graphs. There are many 1-extendable graphs, for instance, all 2-connected cubic graphs are 1-extendable (consequence of Petersen's the famous theorem). So the focus is now on the classification of 2-extendable planar graphs. Several authors (e.g., Aldred, Holton, Lou, Plummer *et al.*, see [253; 251; 487; 492]) have discussed the sufficient conditions of 2-extendability in planar graphs and also used extra conditions to insure 2-extendability, for example, connectivity, local connectivity and cyclically connectivity. We list two results here to show the taste to readers.

Theorem 6.3.13 (Plummer [487]). *Every 5-connected planar graph of even order is 2-extendable.*

The above theorem can be strengthened to the following one.

Theorem 6.3.14 (Lou and Yu [395]). *Let G be a 5-connected planar graph of order p . Then G is $(4 - \varepsilon)$ -factor-critical, where $\varepsilon = 0$ or 1 and $\varepsilon \equiv p \pmod{2}$.*

Proof. Suppose that G is not $(4 - \varepsilon)$ -factor-critical. By Theorem 6.1.4, since G is 5-connected, there exists a subset $S \subseteq V(G)$ with $|S| \geq 5 > 4 - \varepsilon$ such that for some $k \geq 1$

$$o(G - S) = |S| - (4 - \varepsilon) + 2k \geq 2. \quad (6.6)$$

We choose S as small as possible subject to (6.6). And let $C_1, C_2, \dots, C_t$ be the odd components of $G - S$, where $t = |S| - (4 - \varepsilon) + 2k$.

We claim that, for each x of S , x is adjacent to at least three of $C_1, C_2, \dots, C_t$. Otherwise, there exists a vertex $x \in S$ which is adjacent to at most two of $C_1, C_2, \dots, C_t$. Let $S' = S - \{x\}$. Then $o(G - S') = |S'| - (4 - \varepsilon) + 2q$ for some $q \geq k$ and $|S| > |S'| \geq 4 - \varepsilon$, which contradicts to the choice of S or the connectivity of G .

Since G is 5-connected, for each component C of $G - S$, C is adjacent to at least five vertices in S . Now we obtain a bipartite graph H with bipartition (S, Y) by deleting all edges in $G[S]$ and contracting each component of $G - S$ to a vertex and deleting the multiple edges. Clearly, H is planar and triangle free. On the other hand, for each vertex v in S , $d_H(v) \geq 3$, and for each vertex u in Y , $d_H(u) \geq 5$. As G is 5-connected, we have $|S| \geq 5$ and $|Y| \geq$

$|S| - (4 - \varepsilon) + 2k \geq 3$. So $|E(H)| \geq \frac{1}{2}(3|S| + 5|Y|)$. Since $|Y| \geq |S| - (4 - \varepsilon) + 2$, we can write $|Y| = |S| - (4 - \varepsilon) + 2 + m$ for some $m \geq 0$. Then

$$|V(H)| = |S| + |Y| = 2|S| - (4 - \varepsilon) + 2 + m$$

and

$$\begin{aligned} |E(H)| &\geq \frac{1}{2}[3|S| + 5(|S| - (4 - \varepsilon) + 2 + m)] \\ &= (4|S| - 2(4 - \varepsilon) + 4 + 2m - 4) - \frac{1}{2}(4 - \varepsilon) + 5 + \frac{m}{2} \\ &> 2(|V(H)| - 2). \end{aligned}$$

But applying Euler's formula to triangle-free planar graphs, we see $|E(H)| \leq 2|V(H)| - 4$, a contradiction. $\square$

Theorem 6.3.15 (Holton and Plummer [253]). *Every 3-connected, cyclically 5-connected cubic planar graph is 2-extendable.*

For factor-criticality, it was shown that a 4-connected planar graph of even order is bicritical.

Next we study the extendability and factor-criticality of graphs embedded on surfaces of higher genus.

If surface Σ is obtained from the sphere by adding g handles (respectively, $\bar{g}$ crosscaps), then Σ is called a *surface of orientable genus $g = g(\Sigma)$* (respectively a *surface of non-orientable genus $\bar{g} = \bar{g}(\Sigma)$*). Denote the surface of orientable genus g (respectively non-orientable genus $\bar{g}$) by S_g (respectively by $N_{\bar{g}}$). For example, S_0 is the sphere, S_1 is the torus, N_1 is the projective plane and N_2 is the Klein bottle.

If a connected graph is embedded in a surface Σ , then the complement of its image relative to Σ is a collection of connected, open sets called *faces*. If each face is homeomorphic to an open disc, then the embedding is said to be a *2-cell embedding*. The orientable genus $g(G)$ of a graph G is the smallest integer h such that G is embeddable in S_h , and any such embedding is called an *orientably minimal embedding*; by analogy, the non-orientable genus $\bar{g}(G)$ of G and a non-orientably minimal embedding of G are defined by N_k ; since $k = 0$ is not allowed, every planar graph has non-orientable genus equal to one. There is a well-known theorem by Youngs (1963) which states that every minimal orientable embedding of a graph G is 2-cell. A more recent result stated that every graph G has a minimal non-orientable embedding which is 2-cell (see Parsons *et al.* (1987)). Now we can state the well-known Euler's formula for surfaces.

Theorem 6.3.16 (Euler's Formula). *Let G be a graph, which is 2-cell embedded on a surface Σ of genus g , having p vertices, q edges and f faces. Then*

$$p - q + f = 2 - 2g.$$

For non-orientable 2-cell embedding of G , then

$$p - q + f = 2 - \bar{g}.$$

For convenience, denote the right hand side of Euler's Formula by χ , which is called *Euler characteristic*. That is, $\chi(\Sigma) = 2 - 2g$ or $2 - \bar{g}$. Let v be any vertex of a graph G minimally embedded on a surface. Define the *Euler contribution* of vertex v to be

$$\Psi(v) = 1 - \frac{d_G(v)}{2} + \sum_{i=1}^{d_G(v)} \frac{1}{f_i},$$

where the sum runs over the face angles at vertex v and f_i denotes the size of the i th face at v . A vertex u is called a *control point* of the embedding if $\Psi(u) \geq \chi(\Sigma)/p$, i.e., the average value of $\chi(\Sigma)$.

Let Σ be a surface, either orientable or non-orientable. Denote $\xi(\Sigma)$ the smallest integer such that no graph G embeddable in surface Σ is $\xi(\Sigma)$ -extendable. For instance, a cube is 2-extendable but no planar graph is 3-extendable, so $\xi(S_0) = 3$. Dean [159] showed that $\xi(N_1) = 3$, $\xi(S_1) = \xi(N_2) = 4$.

Using Euler contribution, Plummer [482] provided an upper bound for $\xi(\Sigma)$ and later Dean [159] determined the value of $\xi(\Sigma)$ completely.

Theorem 6.3.17 (Dean [159]).

- (a) If Σ is an orientable surface of genus $g > 0$, then $\xi(\Sigma) = 2 + \lfloor 2\sqrt{g} \rfloor$;
- (b) if Σ is a non-orientable surface of genus $\bar{g} > 0$, then $\xi(\Sigma) = 2 + \lfloor \sqrt{2\bar{g}} \rfloor$.

Even though, from the above theorem, the extendability function $\xi(\Sigma)$ is an increasing function of g (and of $\bar{g}$), but actually there are not many graphs with high extendability, even 4-extendability, regardless of the high genus. This fact is discovered recently by Aldred *et al.* (2008). The proof given here is due to T. Wang (2006) in an unpublished manuscript.

Theorem 6.3.18 (Aldred, Kawarabayashi and Plummer [21]). *Let G be a connected graph of orientable genus g (respectively non-orientable genus $\bar{g}$) on n vertices. Then if $n \geq 8g - 7$ (respectively $n \geq 4\bar{g} - 7$), G is not 4-extendable.*

Proof. Without loss of generality, we assume $n \geq 9$. Suppose that, by contrary, G is 4-extendable. Suppose graph G have been minimally embedded on a surface Σ . By the simple counting argument using Euler's formula, there exists a control point v on the surface Σ . If there are x triangular faces at v and $d(v) = d$, then $G[N(v)]$ contains a matching of size $\lceil x/2 \rceil$ and thus G is not $(d - \lceil x/2 \rceil)$ -extendable. So we have $d - \lceil x/2 \rceil > 4$, i.e., $d - \lceil x/2 \rceil \geq 5$, or $d \geq 5 + \lceil x/2 \rceil \geq 5 + x/2$.

On the other hand, from the definition of control point, then

$$d/2 \leq 1 - \chi(\Sigma)/n + \sum_{i=1}^{d(v)} \frac{1}{f_i} \leq 1 - \chi(\Sigma)/n + x/3 + (d-x)/4$$

or

$$d \leq 4 - 4\chi(\Sigma)/n + x/3.$$

Combining with $d \geq 5 + x/2$, we have

$$1 \leq 1 + x/6 \leq -4\chi(\Sigma)/n$$

or

$$n \leq -4\chi(\Sigma).$$

If Σ is orientable, then $\chi = 2 - 2g$, or $n \leq 8g - 8$, a contradiction. If Σ is non-orientable, then $\chi = 2 - \bar{g}$, or $n \leq 4\bar{g} - 8$, a contradiction again.

We complete the proof. $\square$

Aldred *et al.* [21] also constructed infinitely many graphs with genus g or $\bar{g}$ which are 3-extendable.

Su and Zhang [534] obtained analogue results regarding factor-criticality for graphs with high genus. Denote $\rho(\Sigma)$ the smallest integer h such that no graph embeddable in Σ is h -factor-critical. For the case of Klein bottle N_2 , $\rho(N_2) = 6$ was determined by Plummer and Zha [496].

Theorem 6.3.19 (Su and Zhang [534]). *The factor-criticality of a graph is*

- (a) $\rho(S_0) = 5$;
- (b) if S_h is an orientable surface of genus $h > 0$, then $\rho(S_h) = \lfloor \frac{5+\sqrt{1+48h}}{2} \rfloor$;
- (c) if N_k is non-orientable of genus $k \geq 1$ and $k \neq 2$, then $\rho(N_k) = \lfloor \frac{5+\sqrt{1+24k}}{2} \rfloor$.

6.3.4 Matching extension, degree sum and closure operations

One of the most commonly used graphic parameter as a sufficient condition for the study of graph property is probably minimum degree. In this section, we study the extendability in terms of minimum degree, degree sum and other related conditions.

Theorem 6.3.20. *Suppose $1 \leq k \leq n-1$ and $|V(G)| = 2n$.*

- (a) *If $\delta(G) \geq n+k$, then G is k -extendable.*

- (b) Let $G = (X, Y)$ be a bipartite graph with $|X| = |Y|$. If $\delta(G) \geq (n + k)/2$, then G is k -extendable.

In Theorem 6.3.20, part (a) was proved by Plummer [478] and (b) is due to Ananchuen and Caccetta [41]. Next we turn to degree sum and neighborhood union.

Theorem 6.3.21 (Plummer [484]). *Suppose G is a r -connected graph of order $2m$ and n is an integer, $0 \leq n < m$. If there exists an integer t , $2 \leq t \leq r - 2n + 2$, such that $\sum_{i=1}^t d_G(w_i) \geq t(m + n - 1) + 1$ for any independent set $I = \{w_1, w_2, \dots, w_t\}$ of order t , then G is $2n$ -factor-critical. In particular, under the same conditions, G is n -extendable.*

Proof. We consider $t = 1$ first. In this case, the degree-sum condition becomes $d_G(u) \geq m + n$ for any $u \in V(G)$. This is exactly Theorem 6.3.20 (a).

Now we assume that $t \geq 2$. The proof proceeds by induction on n .

For $n = 0$, it is known that the degree-sum condition in this case implies that G has a Hamilton path and thus G has a 1-factor.

Suppose the theorem holds for $n - 1$, i.e., G is $(2n - 2)$ -factor-critical. We shall show that G is $2n$ -factor-critical.

If G is not $2n$ -factor-critical, by Theorem 6.1.4, there exists a vertex-set $S \subseteq V(G)$ with $|S| \geq 2n$ such that $o(G - S) > |S| - 2n$. By induction hypothesis, G is $(2n - 2)$ -factor-critical and thus $o(G - S) \leq |S| - (2n - 2)$. The parity argument yields $o(G - S) = |S| - 2n + 2$. Let $h = |S| - 2n + 2$.

Now let t , $2 \leq t \leq r - 2n + 2$, be as given in the hypothesis. Since G is r -connected, we have $|S| \geq r$ or $h = |S| - 2n + 2 \geq r - 2n + 2 \geq t$. Denote the odd components of $G - S$ by $C_1, C_2, \dots, C_h$. Choose $w_i \in V(C_i)$ for $i = 1, 2, \dots, t$, then $\{w_1, w_2, \dots, w_t\}$ is an independent set and

$$\sum_{i=1}^t d(w_i) \leq \sum_{i=1}^t (|V(C_i)| - 1) + t|S| = \sum_{i=1}^t |V(C_i)| + t(|S| - 1). \quad (6.7)$$

On the other hand, $\sum_{i=1}^h |V(C_i)| + |S| \leq 2m$, so

$$\sum_{i=1}^t |V(C_i)| \leq 2m - |S| - \sum_{i=t+1}^h |V(C_i)|. \quad (6.8)$$

So substituting (6.8) into (6.7), we obtain

$$\sum_{i=1}^t d(w_i) \leq 2m - |S| - \sum_{i=t+1}^h |V(C_i)| + t(|S| - 1).$$

But $\sum_{i=t+1}^h |V(C_i)| \geq h - t$, so

$$- \sum_{i=t+1}^h |V(C_i)| \leq t - h.$$

Hence

$$\sum_{i=1}^t d(w_i) \leq 2m - |S| + t - h + t(|S| - 1) = 2m + (t - 1)|S| - h. \quad (6.9)$$

But $h + |S| \leq 2m$ and $h = |S| - 2n + 2$, thus

$$|S| \leq \frac{2m + 2n - 2}{2}. \quad (6.10)$$

So substituting (6.10) into (6.9) (and since $t \geq 2$), we have

$$\sum_{i=1}^t d(w_i) \leq t(m + n - 1)$$

a contradiction to the hypothesis and we complete the induction. $\square$

To relate the above result to the classic Ore's condition, we include the extendability version of the result for the special case $t = 2$ as a corollary here.

Corollary 6.3.1. *Let G be a graph of order $2m$ and k be an integer, $1 \leq k < m$. Suppose that for all pairs of non-adjacent vertices u and v in G , $d_G(u) + d_G(v) \geq 2m + 2k - 1$. Then G is k -extendable.*

Favaron [202] extended Theorem 6.3.21 to n -factor-critical graphs by modifying the proof of Theorem 6.3.21. Actually, the proof technique in the above theorem can be easily extended to deal with neighborhood union.

Theorem 6.3.22 (Plummer [484]). *Suppose G is a r -connected graph of order $2m$ and n is an integer, $0 \leq n < m$. If there exists an integer t , $1 \leq t \leq r - 2n + 2$, such that $|\cup_{i=1}^t N_G(w_i)| \geq 2m - r + 2n - 1$ for any independent set $I = \{w_1, w_2, \dots, w_t\}$ of order t , then G is $2n$ -factor-critical. In particular, under the same conditions, G is n -extendable.*

Closure operation has been a successful tool for the study of Hamiltonian cycle problem initiated by Bondy and Chvátal. Adopted its success, Plummer and Saito [494] studied the relation between various closures and n -factor-criticality. They obtained a necessary and sufficient condition for a graph to be n -factor-critical in terms of a Bondy-Chvátal-type closure and also a closure based on neighborhood unions.

Theorem 6.3.23 (Plummer and Saito [494]). *Let G be a graph of order p and let x and y be a pair of nonadjacent vertices of G with $d_G(x) + d_G(y) \geq p + n - 1$. Then G is n -factor-critical if and only if $G \cup xy$ is n -factor-critical.*

Proof. If G is n -factor-critical, clearly $G \cup xy$ is also n -factor-critical.

On the other hand, assume $G \cup xy$ is n -factor-critical but G is not n -factor-critical. Then, by Theorem 6.1.4, there exists a subset $S \subseteq V(G)$ with $|S| \geq n$ such that $o(G - S) > |S| - n$ and $o((G \cup xy) - S) \leq |S| - n$. As G is n -factor-critical, $p \equiv n \pmod{2}$ and thus $o(G - S) + |S| \equiv p \pmod{2}$. But $o((G \cup xy) - S) \geq o(G - S) - 2$, so we have $o(G - S) = |S| - n + 2$ and $o((G \cup xy) - S) = |S| - n$. This also implies that x and y , respectively, belong to two different odd components C_1, C_2 of $G - S$. Then $d_G(x) + d_G(y) \leq |V(C_1)| - 1 + |V(C_2)| - 1 + 2|S|$. On the other hand, by the hypothesis and since each of the $|S| - n$ odd components which is different from C_1 and C_2 of $G - S$ contains at least one vertex,

$$d_G(x) + d_G(y) \geq |S| + |V(C_1)| + |V(C_2)| + (|S| - n) + n - 1.$$

This yields $2|S| + |V(C_1)| + |V(C_2)| - 1 \leq 2|S| + |V(C_1)| + |V(C_2)| - 2$, a contradiction. $\square$

By similar arguments as those in the proof of the above theorem, we can show that if $G \cup xy$ is k -extendable for a pair of nonadjacent vertices x and y with $d_G(x) + d_G(y) \geq |V(G)| + 2k - 1$, then G is k -extendable. But the reverse of this statement is not true. Consider an example, $G = (k + 1)K_1 + (K_{2m+k} \cup K_{2m+2k+1})$. Then it is easy to verify that G is k -extendable, but $G \cup xy$ is not for $\{x, y\} \subseteq V((k + 1)K_1)$.

The similar closure operation can be introduced for neighborhood unions.

Theorem 6.3.24 (Plummer and Saito [494]). *Let G be an r -connected graph of order p , and let x and y be a pair of nonadjacent vertices of G with $|N_G(x) \cup N_G(y)| \geq p - r + n - 1$. Then*

- (a) G is n -factor-critical if and only if $G \cup xy$ is n -factor-critical;
- (b) G is $(\frac{n}{2})$ -extendable if and only if $G \cup xy$ is $(\frac{n}{2})$ -extendable.

Recently, Ryjáček introduced a new closure for claw-free graphs. For a vertex $v \in V(G)$, if $G[N_G(v)]$ is an r -connected subgraph, then v is called a *locally r -connected* vertex. The *local completion* of G at v is the operation of replacing the induced subgraph $G[N_G(v)]$ by the complete graph K_m , where $m = |N_G(v)|$. Ryjáček r -closure $C_r(G)$ is a graph obtained from a claw-free graph G by iteratively performing local completion at locally r -connected vertices until no more edges can be added.

Plummer and Saito [494] proved the following theorems by using the local completion operation.

Theorem 6.3.25 (Plummer and Saito [494]). *Let G be a claw-free graph and x a locally n -connected vertex. Let G^* be the graph obtained from G by local completion at x in G . Then G is n -factor-critical if and only if G^* is n -factor-critical.*

Theorem 6.3.26 (Plummer and Saito [494]). *Let G be a claw-free graph. Then G is n -factor-critical if and only if $C_n(G)$ is n -factor-critical.*

Nishimura [462] generalized the above results by using a neighborhood condition. Recall that $N_G[v] = N_G(v) \cup \{v\}$.

Theorem 6.3.27 (Nishimura [462]). *Let G be a graph and x a locally n -connected vertex. Let u and v be a pair of vertices in $V(G) - \{x\}$ such that $uv \notin E(G)$, $x \in N_G(u) \cap N_G(v)$, and $N_G(x) \subset N_G[u] \cup N_G[v]$. Then G is n -factor-critical if and only if $G \cup uv$ is n -factor-critical.*

Proof. The necessity is obvious. We prove sufficiency by contradiction. Suppose $G \cup uv$ is n -factor-critical but G is not n -factor-critical. Then, by Theorem 6.1.4, there exists a subset $S \subseteq V(G)$ with $|S| \geq n$ such that $o(G - S) > |S| - n \geq o((G \cup uv) - S)$. Notice that $|V(G)| \equiv n \pmod{2}$ since $G \cup uv$ is n -factor-critical. Since $o(G - S) + |S| \equiv |V(G)| \pmod{2}$ and $o((G \cup uv) - S) \geq o(G - S) - 2$, we have $o(G - S) - 2 = o((G \cup uv) - S) = |S| - n$. Therefore, we may assume $u \in C_1$ and $v \in C_2$, where C_1 and C_2 are odd components of $G - S$. Let $C_3, \dots, C_{|S|-n+2}$ be the other odd components of $G - S$. These components are also odd components of $(G \cup uv) - S$. Since $E_G(C_1, C_2) = \emptyset$ and $x \in N_G(u) \cap N_G(v)$, we may assume $x \in S$.

Case 1. $|S| = n$. In this case, two vertices $u, v \in N_G(x)$ are separated by $N_G(x) \cap S$ in $G[N_G(x)]$. Since $|N_G(x) \cap S| < n$, this contradicts the assumption that x is locally n -connected.

Case 2. $|S| > n$. Choose a vertex subset $T \subset S - \{x\}$ with $|T| = n$. Since $G \cup uv$ is n -factor-critical, $(G \cup uv) - T$ has a 1-factor so that every vertex of $S - T$ is matched with a vertex of distinct components $C_3, \dots, C_{|S|-n+2}$. In particular, we may assume x is matched with a vertex w of C_3 . However, since $N_G(x) \subset N_G[u] \cup N_G[v]$, w is adjacent to u or v in G . This is impossible since $E_G(C_1 \cup C_2, C_3) = \emptyset$, which completes the proof. $\square$

Lou [383] used other two types of neighborhood conditions as sufficient conditions for the k -extendability. For two vertices u and v of a graph G , let $N(u, v) = N_G(u) \cap N_G(v)$ and $n(u, v) = |N(u, v)|$. For two vertices u and v with $d_G(u, v) = 2$, let

$$I_{u,v}(w) = \max\{|S| \mid w \in N(u, v), S \text{ is an independent set containing } u \text{ and } v \text{ in } G[N_G[w]]\},$$

we define *divergence* $\alpha^*(u, v) = \max\{I_{u,v}(w) \mid w \in N(u, v)\}$. This parameter was used as a sufficient condition for the existence of Hamiltonian cycles in the literatures.

Theorem 6.3.28 (Lou [383]). *Let G be a connected graph of even order. If $n(u, v) \geq \alpha^*(u, v) + 2k - 1$ for any two vertices u and v with $d_G(u, v) = 2$, then G is $2k$ -factor-critical. In particular, under the same conditions, G is k -extendable.*

Theorem 6.3.29 (Lou [383]). *Let G be a connected graph of even order. For any vertex $v \in V(G)$, denote $G_v = G[\{v\} \cup N_G(v)]$. If*

$$d_{G_v}(x) + d_{G_v}(y) \geq d_G(v) + 2k$$

for any two nonadjacent vertices $x, y \in V(G_v)$, then G is $2k$ -factor-critical.

Remark 6.3.2. As seen in the above, we have obtained the sufficient conditions of k -extendability in terms of neighborhood union, degree sum, divergence. Moreover, there are more such conditions, such as modified Chvátal-Erdős type condition (Lou (1994)), cyclic connectivity condition (Holton and Plummer (1988)), etc. It is interesting to note that these conditions are also used as sufficient conditions for Hamiltonian cycles. Although the concepts of k -extendability and Hamiltonian cycle are not closely related, it is observed that many conditions that ensure the existence of Hamiltonian cycles also become sufficient condition for k -extendability. There are still many other Hamiltonian cycle friendly conditions open for further investigation.

6.3.5 Matching extension and product of graphs

The product of graphs is an interesting operation to investigate. Let us start with the most common product, Cartesian product. Let G_1 and G_2 be two graphs. The *Cartesian product* $G_1 \times G_2$ is a graph with vertex-set $V(G_1 \times G_2) = \{(x_1, x_2) \mid x_1 \in V(G_1), x_2 \in V(G_2)\}$ and two vertices (x_1, x_2) and (y_1, y_2) are adjacent if and only if $x_1 = y_1$ and $x_2 y_2 \in E(G_2)$, or $x_2 = y_2$ and $x_1 y_1 \in E(G_1)$. For example, the cube is $C_4 \times K_2$.

We investigate the extendability and factor-criticality in the product of graphs. A special but important case of Cartesian product is the following result, which is used many times in the proofs of other theorems.

Theorem 6.3.30 (Györi and Plummer [234]; Liu and Yu [375]). *If G is a k -extendable graph, then $G \times K_2$ is $(k + 1)$ -extendable.*

Proof. Let $K_2 = uv$ and $H = G \times K_2$. We denote the subgraphs induced by vertices $V(G) \times \{u\}$ and $V(G) \times \{v\}$ in H by G_u and G_v , respectively, each of them is isomorphic to G .

Let $E(G_u, G_v)$ denote the edges between $V(G) \times \{u\}$ and $V(G) \times \{v\}$ in H . Then $E(G_u, G_v)$ is a 1-factor between G_u and G_v . Let P_{uv} be a bijection

from G_u to G_v with $P_{uv}((x, u)) = (x, v)$ for every $x \in V(G)$. Hence, the projection P_{uv} is an isomorphism from G_u to G_v .

Let $M = \{e_1, \dots, e_r, e'_1, \dots, e'_s, u_1 u'_1, \dots, u_h u'_h\}$ be any $(k+1)$ -matching in H , where $r + s + h = k + 1$, $e_i \in E(G_v)$ for $1 \leq i \leq r$, $e'_j \in E(G_u)$ for $1 \leq j \leq s$, and $u_t \in V(G_v)$, $u'_t \in V(G_u)$ for $1 \leq t \leq h$.

We project $\{e'_1, \dots, e'_s\}$ to G_v by using P_{uv} such that

- (a) if $P_{uv}(e'_j)$ is incident to just one edge of $e_1, e_2, \dots, e_r$, then we delete this edge but keep the end-vertex which is not incident to any edge of $e_1, e_2, \dots, e_r$, and
- (b) if $P_{uv}(e'_j)$ is incident to two of $e_1, e_2, \dots, e_r$, then delete $P_{uv}(e'_j)$.

Then we obtain a matching $M' = \{e_1, e_2, \dots, e_r, e_{r+1}, \dots, e_p\}$ and a vertex set $U' = \{u_1, u_2, \dots, u_h, u_{h+1}, \dots, u_q\}$, where $p + q \leq k + 1$, and U' and M' are independent.

Case 1. $q = 0$. Then $r + s = k + 1$. If $s = 0$ or $r = 0$, say $s = 0$, then $M \subseteq G_v$. Let $M^* = M \cup P_{vu}(M) \cup \{x P_{vu}(x) \mid x \in V(G_v) - V(M)\}$. Then M^* is a 1-factor of H containing M . So suppose $r > 0$ and $s > 0$, then $r \leq k$ and $s \leq k$. But G is k -extendable, by Theorem 6.1.11 (1), there exist 1-factors F_1 and F_2 in G_v and G_u which contain matchings $\{e_1, \dots, e_r\}$ and $\{e'_1, \dots, e'_s\}$, respectively. Hence $F_1 \cup F_2$ is a 1-factor of H as required.

Case 2. $q > 0$. By Theorem 7.3.6, we see that G_v is $[p, q]$ -extendable. Let M_1^* be an extension of (M', U') in G_v . Let $u_1 v_1, \dots, u_q v_q \in M_1^*$. Then $\{e'_1, \dots, e'_s, u'_1 v'_1, \dots, u'_q v'_q\}$ is a matching of G_u , where $s + q < k$ and $u'_i = P_{vu}(u_i)$. Hence $\{e'_1, \dots, e'_s, u'_1 v'_1, \dots, u'_q v'_q\}$ can be extended to a 1-factor M_2^* in G_u . Now let

$$M^* =$$

$$\{u_i u'_i, v_i v'_i \mid 1 \leq i \leq h\} \cup (M_1^* - \{u_i v_i \mid 1 \leq i \leq h\}) \cup (M_2^* - \{u'_i v'_i \mid 1 \leq i \leq h\}).$$

Then M^* is a 1-factor of H and $M \subseteq M^*$. □

There is an interesting special case of the above result. Consider the n -dimensional cube Q_n , which is defined as $Q_n = Q_{n-1} \times K_2$ and $Q_2 = C_4$. Note that Q_n has 2^n vertices and $n2^{n-1}$ edges and is n -regular.

Corollary 6.3.2. *The n -cube Q_n is $(n-1)$ -extendable.*

Theorem 6.3.30 can be generalized a bit more. The proof is based on induction and use the above theorem as an initial induction step.

Theorem 6.3.31 (Liu and Yu [375]). *If G is a k -extendable graph and H is a connected graph, then $G \times H$ is $(k+1)$ -extendable.*

Use the above theorem, we are able to prove the following more general result. However, most proofs of theorems in section are quite long. So we omit it.

Theorem 6.3.32 (Györi and Plummer [234]). *Let G_i be a k_i -extendable graph and k_i be a non-negative integer ($i = 1, 2$). Then $G_1 \times G_2$ is $(k_1 + k_2 + 1)$ -extendable.*

For factor-criticality, Wu, Yang and Yu (2007) recently obtained the following theorem.

Theorem 6.3.33 (Wu, Yang and Yu [591]). *Let G_1 be an m -factor-critical graph and G_2 an n -factor-critical graph. Then $G_1 \times G_2$ is $(m + n + \varepsilon)$ -factor-critical, where $\varepsilon = 0$, if both of m and n are even; $\varepsilon = 1$, otherwise.*

Note that all three theorems above are sharp.

The *strong product* $G_1 \otimes G_2$ of two graphs G_1 and G_2 has vertex set $V(G_1) \times V(G_2)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent if (i) $u_1 = u_2$ and v_1 is adjacent to v_2 ; or (ii) u_1 is adjacent to u_2 and $v_1 = v_2$; or (iii) u_1 is adjacent to u_2 and v_1 is adjacent to v_2 . For example, $K_2 \otimes K_2 = K_4$.

Györi and Imrich [233] studied extendability of strong product and obtained the following nice result.

Theorem 6.3.34 (Györi and Imrich [233]). *Let $G_1 \otimes G_2$ be the strong product of a k -extendable graph G_1 and an l -extendable graph G_2 . Then $G_1 \otimes G_2$ is $2\lfloor (k+1)(l+1)/2 \rfloor$ -factor-critical. In particular, $G_1 \otimes G_2$ is $\lfloor (k+1)(l+1)/2 \rfloor$ -extendable.*

The extendability in Theorem 6.3.34 is not best possible. In fact, Györi and Imrich [233] posed the following conjecture.

Conjecture 6.3.1 (Györi and Imrich (2001)). *Let $G_1 \otimes G_2$ be the strong product of a k -extendable graph G_1 and an l -extendable graph G_2 . Then $G_1 \otimes G_2$ is $(2\lfloor (k+2)(l+2)/2 \rfloor - 2)$ -factor-critical.*

6.3.6 Matching extension and other parameters

In this section, we include some results on the extendability which can not be classified in the previous sections.

We first discuss the factor-criticality of power graphs. The r th *power* of a graph G , G^r , is the graph with vertex set $V(G)$ and edge set $\{uv \mid d_G(u, v) \leq r\}$.

Theorem 6.3.35 (Shavo [524]). *Let G be a graph of order $2n$ and $0 \leq k \leq n - 1$. Then, if G^r is k -extendable, so is G^{r+1} .*

Theorem 6.3.36 (Lou and Yu [395]). *Let G be a graph of order p and n an integer such that $0 \leq n \leq p - 2$ and $n \equiv p \pmod{2}$.*

- (a) *If G is h -connected and $h > \lfloor \frac{n}{2} \rfloor$, then G^r is n -factor-critical for $r \geq 2$;*
- (b) *if G is h -connected and $1 \leq h \leq \lfloor \frac{n}{2} \rfloor$, then G^r is n -factor-critical, for $r \geq n - 2h + 3$.*

Proof. Suppose that G^r is not n -factor-critical. By Theorem 6.1.4, there is a subset $S \subseteq V(G)$ with $|S| \geq n$ such that $o(G^r - S) = |S| - n + 2m$ for some $m \geq 1$. Let $S_1 = S - \{v_1, v_2, \dots, v_n\}$, where $v_1, v_2, \dots, v_n$ are any n vertices in S . Then $o(G^r - S) = |S_1| + 2m$.

(a) For the case of $h > \lfloor \frac{n}{2} \rfloor$, as G is h -connected, each component of $G^r - S$ is adjacent in G to at least h vertices in S . Suppose that no two odd components of $G^r - S$ in G have a common neighbor in S . Then there are at least $(|S_1| + 2m)h$ vertices in S . But S has only $|S| = |S_1| + n < (|S_1| + 2m)h$ vertices, a contradiction. So at least two odd components, say C_1 and C_2 , have a common neighbor v in S . Then there is a vertex u in C_1 and a vertex w in C_2 such that $uv \in E(G)$ and $wv \in E(G)$. In G^r , u and w are adjacent. So u and w are in the same component of $G^r - S$, a contradiction to the fact that C_1 and C_2 are different components of $G^r - S$.

(b) For the case of $1 \leq h \leq \lfloor \frac{n}{2} \rfloor$, let $C_1, C_2, \dots, C_t$ be the components of $G^r - S$ and let N_i be the set of vertices in S adjacent to vertices of C_i in G . Since G is h -connected, each N_i contains at least h vertices. Furthermore, N_i 's are pairwise disjoint. Otherwise, a component C_i contains a vertex u that is distance two from a vertex v in another component C_j . But then u and v would be in the same component of $G^r - S$. Because G is connected, there exists a path P in G from a vertex w_i in N_i to a vertex w_j in N_j ($i \neq j$). Choose $\hat{P}$ to be such a path with the minimum length among all the path P 's. Then $\hat{P}$ is contained in S and none of the internal vertices of $\hat{P}$ is in N_l ($1 \leq l \leq t$). Since $|S| = |S_1| + n$ and $t \geq |S_1| + 2m$, the order of $\hat{P}$ is at most $|S_1| + n - h(|S_1| + 2m) + 2 \leq |S_1| + n - h(|S_1| + 2) + 2 = n - 2h - |S_1|(h - 1) + 2 \leq n - 2h + 2$. There is a vertex z_i in C_i and a vertex z_j in C_j adjacent to w_i and w_j , respectively. Then $z_i \hat{P} z_j$ is a path of length at most $n - 2h + 3$. So z_i and z_j are adjacent in G^r , which contradicts to the fact that C_i and C_j are different components of $G^r - S$ again. $\square$

The above result is a generalization of a theorem by Holton, Lou and McAvaney [250] about extendability. In the same paper, they also studied the extendability of line graphs and total graphs.

Theorem 6.3.37 (Holton, Lou and McAvaney [250]). *Let G be a graph of even order. Then the line graph $L(G)$ is k -extendable if and only if, for any collection $Q_1, Q_2, \dots, Q_k$ of edge disjoint P_3 's in G , $G - E(Q_1) - \dots - E(Q_k)$ has no odd components.*

Corollary 6.3.3. *If G is a $(k + 2)$ -connected graph of even order, then $L(G)$ is k -extendable.*

Theorem 6.3.38 (Holton, Lou and McAvaney [250]). *If total graph $T(G)$ is $(2k+1)$ -connected and has even order, then $T(G)$ is k -extendable.*

We can also use the independent number $\delta(G)$ to relate to extendability. Below are results related to independent number.

Theorem 6.3.39 (Ananchuan and Caccetta [42]). *Let G be a graph of order $2m$ and $\delta(G) \geq m+k-1$, where k is a positive integer such that $m/2 \leq k \leq m-2$ and $m-k$ is even. Then G is k -extendable if and only if $\alpha(G) \leq m-k$.*

Theorem 6.3.40 (Favaron [202]). *Let r and n be nonnegative integers. Then every r -connected graph G with $\alpha(G) \leq r-n+1$ and $|V(G)| \equiv n \pmod{2}$ is n -factor-critical.*

Theorem 6.3.41 (Maschlanka and Volkmann [421]). *Let G be a k -extendable, nonbipartite graph of order n . Then $\alpha(G) + k \leq \frac{n}{2}$.*

The famous Chvátal-Erdős condition for Hamilton cycle can also inspire us to obtain a similar sufficient condition for k -extendable graphs.

Theorem 6.3.42 (Maschlanka and Volkmann [421]). *Let G be a graph of even order $2n$ and k a nonnegative integer with $k < n$. If $\kappa(G) \geq \alpha(G) + 2k - 1$, then G is k -extendable.*

6.4 Extendability of symmetric graphs

In this section, we investigate the extendability of symmetric graphs. In particular, we consider the generalized Petersen graphs, Cayley graphs, transitive graphs, etc.

Let Γ be a group and S a subset of Γ such that the identity element $1 \notin S$ (i.e., $S \subseteq \Gamma - \{1\}$) and $x^{-1} \in S$ for each $x \in S$. The *Cayley graph* $G(\Gamma; S)$ on a group Γ has the elements of Γ as its vertices and edges joining g and gs for all $g \in \Gamma$ and $s \in S$. We call S the *generating set*. It is well-known that every Cayley graph is vertex-transitive. Moreover, if S generates Γ , then $G(\Gamma; S)$ is connected; otherwise it consists of disjoint copies of $G(\langle S \rangle; S)$.

For $S \subseteq \Gamma$, we denote by $\langle S \rangle$ the subgroup of Γ induced by S . When the group is cyclic, or $G(\Gamma; S) \cong Z_m$ for some m , the Cayley graph $G(\Gamma; S)$ is called a *circulant* and is denoted by $Z_m(S)$. The *dihedral group* D_n is a group which generated by two elements ρ and τ , where $\rho^n = \tau^2 = 1$ and $\tau\rho\tau = \rho^{-1}$. From the relations of $\rho^n = \tau^2 = 1$ and $\tau\rho\tau = \rho^{-1}$, we can easily obtain $(\rho^i\tau)^2 = 1$ and $\rho^i\tau\rho^{-j} = \tau\rho^{-(i+j)} = \rho^{i+j}\tau$. It is not hard to see that D_n has a cyclic subgroup $\langle \rho \rangle$ of index 2 which is isomorphic to Z_n . Moreover, $D_n = \langle \rho \rangle \cup \langle \rho \rangle \tau$.

The *dicyclic group* Q_{2n} is a group generated by two elements a and x , where $a^{2n} = 1$, $x^2 = a^n$ and $x^{-1}ax = a^{-1}$. We denote $\{mx \mid m \in \langle a \rangle\}$ by $\langle a \rangle x$. From the relations $a^{2n} = 1$, $x^2 = a^n$ and $x^{-1}ax = a^{-1}$, we can easily obtain $(a^i x)^{-1} = a^{i+n} x$, $xa^i x = a^{n-i}$ and $a^i x = xa^{2n-i}$. Once again, it is not hard to see that Q_{2n} has a cyclic subgroup $\langle a \rangle$ of index 2, which is isomorphic to Z_{2n} . Moreover, $Q_{2n} = \langle a \rangle \cup \langle a \rangle x$, and $|Q_{2n}| = 4n$.

The most general results on 1-extendability of symmetric graphs are given by Lovász and Plummer [412].

Theorem 6.4.1 (Lovász and Plummer [412]). *Let G be a connected graph of order n . If G is vertex-transitive, then*

- (a) *if n is odd, then G is factor-critical;*
- (b) *if n is even, then G is either 1-extendable bipartite or bicritical.*

Corollary 6.4.1. *Every vertex-transitive graph of even order is 1-extendable.*

Chen and Quimpo [143] showed that every Cayley graph of even order is 1-extendable. Since Cayley graphs are vertex-transitive, Chen and Quimpo's result is an immediate corollary of the above theorem.

Stong [532] studied 1-factorizability of Cayley graphs on various groups (e.g., abelian group, dihedral group, dicyclic group and any group of order 2^k). Note that 1-factorizability implies 1-extendability.

For 2-extendability, Chan, Chen and Yu [129] classified Cayley graphs on Abelian group completely.

Theorem 6.4.2 (Chan, Chen and Yu [129]). *Let $G(\Gamma; S)$ be a Cayley graph on an Abelian group Γ of even order. Then $G(\Gamma; S)$ is 2-extendable if and only if it is not isomorphic to one of following circulants:*

- (a) $Z_{2n}(1, 2n-1)$, $n \geq 3$;
- (b) $Z_{2n}(1, 2, 2n-1, 2n-2)$, $n \geq 3$;
- (c) $Z_{4n}(1, 4n-1, 2n)$, $n \geq 2$;
- (d) $Z_{4n+2}(2, 4n, 2n+1)$, $n \geq 1$;
- (e) $Z_{4n+2}(1, 4n+1, 2n, 2n+2)$, $n \geq 1$.

Proof. Before we proceed to the proof, let us mention a few facts which can be verified easily (but a bit tedious):

Fact 1. $Z_{2n}(1, 2n-1, n)$ ($n > 2$) is 2-extendable if and only if n is odd;

Fact 2. $Z_{4n+2}(1, 4n+1, k, 4n+2-k)$ ($n \geq 1$ and $k < 2n+1$) is 2-extendable if and only if $k \neq 1, 2, 2n$;

Fact 3. $Z_{4n}(1, 4n-1, k, 4n-k)$ ($n \geq 2$ and $1 \leq k \leq 2n$) is 2-extendable if and only if $k \neq 1, 2, 2n$.

Fact 4. $Z_{2n}(1, 2n-1, 2, 2n-2, n-1, n+1)$ ($n \geq 2$ and $n \neq 3$) is 2-extendable.

Assume that $G(\Gamma; S)$ is isomorphic to one of (a)–(e). It is not hard to verify that each of (a)–(e) is not 2-extendable. In fact, $Z_{2n}(1, 2n-1)$ (i.e., an even cycle) is 2-extendable if and only if $n = 2$; the edges 01 and 34 of $Z_{2n}(1, 2, 2n-1, 2n-2)$ can not be extended to a 1-factor; the graphs in (c) are not 2-extendable by Fact 1; the edges 02 and 13 of $Z_{4n+2}(2, 4n, 2n+1)$ can not be extended to a 1-factor; finally, the graphs in (e) are not 2-extendable by Fact 2.

Conversely, assume that $G(\Gamma; S)$ is not isomorphic to any of graphs (a)–(e). We shall prove that $G(\Gamma; S)$ is 2-extendable.

If $G(\Gamma; S)$ is regular of degree 2, then it must be a 4-cycle and is so 2-extendable.

If $G(\Gamma; S)$ is regular of degree 3, let $S = \{a, b, c\}$. If a, b and c are of order 2, then $G(\Gamma; S)$ is isomorphic to the complete graph K_4 or the cube $C_4 \times P_2$ and is so 2-extendable. Otherwise, we may assume that $a + b = 0$ and $c + c = 0$. If $c \notin \langle a \rangle$, then $G(\Gamma; S) \cong C_m \times P_2$ where m is the order of the element a . By the hypothesis, m must be even. From Theorem 6.3.30, $C_m \times P_2$ is 2-extendable. On the other hand, if $c \in \langle a \rangle$, then a must be of even order $2n$ and $c = na$. Hence $G(\Gamma; S) \cong Z_{2n}(1, 2n-1, n)$ and n is odd. Hence $G(\Gamma; S)$ is 2-extendable by Fact 1.

If $G(\Gamma; S)$ is regular of degree 4, let $S = \{a, b, c, d\}$. Again if a, b, c and d are of order 2, then $G(\Gamma; S)$ is isomorphic to $K_4 \times P_2$, or the complete bipartite graph $K_{4,4}$, or $C_4 \times C_4$ and is so 2-extendable. If a, b are of order 2 and $c + d = 0$, let e_1 and e_2 be two independent edges of $G(\Gamma; S)$, we assume that e_1 and e_2 are the same type (for the case of e_1 and e_2 being different types, it can be discussed similarly but much simpler). If they are of type a or b (say a), then the set of all a -edges constitutes a required 1-factor. Hence, assume that they are of type c . If $a \in \langle c \rangle$, then c must be of even order $2n$ and $b \notin \langle c \rangle$. Hence $G(\Gamma; S)$ has a spanning subgraph which is isomorphic to $C_{2n} \times P_2$ and contains the edges e_1 and e_2 . From Theorem 6.3.30, $C_{2n} \times P_2$ is 2-extendable and thus $G(\Gamma; S)$ has a 1-factor containing e_1 and e_2 . Assume that both a and b are not in $\langle c \rangle$. If c is of even order $2n$, then $\langle c, a \rangle \langle c, -c, a \rangle \cong C_{2n} \times P_2$, which is 2-extendable. Hence there exists a 1-factor M of $\langle c, a \rangle \langle c, -c, a \rangle$ containing e_1 and e_2 , which can then be extended to a 1-factor of $G(\Gamma; S)$. If c is of odd order k , then $G(\Gamma; S)$ is isomorphic to $C_k \times P_2 \times P_2$, which is 2-extendable. Finally, assume that $a + b = 0$ and $c + d = 0$. At least one of a or c (say a) is of even order $2n$. If $c \notin \langle a \rangle$, then for any two independent edges e_1 and e_2 of $G(\Gamma; S)$, there exists a spanning subgraph of $G(\Gamma; S)$ which is isomorphic to $C_{2n} \times P_m$, $m \geq 2$, and contains e_1 and e_2 . So, by Theorem 6.3.31, $G(\Gamma; S)$ has a 1-factor containing e_1 and e_2 . If $c \in \langle a \rangle$, then $c \neq a, na, 2a$ and $c \neq (n-1)a$ if n is odd. In this case, $G(\Gamma; S) \cong Z_{4n}(1, 4n-1, k, 4n-k)$ or $\Gamma \cong Z_{4n+2}(1, 4n+1, k, 4n+2-k)$. By Facts 2 and 3, $G(\Gamma; S)$ is 2-extendable.

Now let $G(\Gamma; S)$ be a regular graph of degree at least 5 and e_1 and e_2 any two independent edges of $G(\Gamma; S)$. Here, we only consider the case when e_1

and e_2 are of the same type (say a), other case can be discussed similarly but much simpler. As $G(\Gamma; S)$ is vertex-transitive, we may assume that $e_1 = 0a$. We need only to consider the following cases:

Case 1. a is of even order $2n$.

If $n = 1$, then all the a -edges of $G(\Gamma; S)$ form a 1-factor of $G(\Gamma; S)$. Hence, let $n \geq 2$. Then the set of all a -edges forms a spanning subgraph of $G(\Gamma; S)$ which is the disjoint union of $2n$ -cycles. If e_1 and e_2 are in two distinct cycles, then clearly they can be extended to a 1-factor of $G(\Gamma; S)$. Hence, let $e_2 = (ta)((t+1)a)$, $1 < t < 2n - 1$. If $\langle a \rangle \neq \Gamma$, then there exists $b \in S$ such that $b \notin \langle a \rangle$. Then $\langle a, b \rangle(a, -a, b, -b)$ has a spanning subgraph H isomorphic to $C_{2n} \times P_m$ containing e_1 and e_2 , for some $m \geq 2$. By Theorem 6.3.31, e_1 and e_2 can be extended to a 1-factor of H , which in turn can be extended to a 1-factor of $G(\Gamma; S)$. Finally, we may assume that $b \in \langle a \rangle$ for all $b \in S$. As $G(\Gamma; S)$ is of degree at least 5, by Fact 4, we may assume that there exists $b \in S$ with $b \neq a, (2n-1)a, 2a, (2n-2)a, (n-1)a$ and $(n+1)a$. If n is odd, then by Facts 1 and 2, the subgraph induced by the set of all a -edges and b -edges is a 2-extendable spanning subgraph of $G(\Gamma; S)$. Hence e_1 and e_2 can be extended to a 1-factor of $G(\Gamma; S)$. Suppose n is even. If there is an element c of S other than $\pm 2a$ and na , then by Fact 3, the subgraph induced by the set of all a -edges and c -edges is a 2-extendable spanning subgraph of $G(\Gamma; S)$. Hence e_1 and e_2 can be extended to a 1-factor of $G(\Gamma; S)$. If such a c does not exist, then $M = \{(ia)((i+1)a) \mid i = 0, 2, \dots, 2n-2\}$ is a 1-factor of $G(\Gamma; S)$ that contains e_1 and e_2 if $e_2 = (ta)((t+1)a)$ with t even. It remains to consider $e_2 = (ta)((t+1)a)$ with t odd. By symmetry we may take $3 \leq t \leq n-1$. Now $M = \{e_1, e_2, (2a)((n+2)a), ((n+1)a)((n+3)a)\}$ may be extended to a 1-factor of $G(\Gamma; S)$ since the set of vertices of $G(\Gamma; S)$ not in M can be partitioned into even paths.

Case 2. a is of odd order $2n+1$.

As Γ is of even order, there exists an element $b \in S$ of even order and so $b \notin \langle a \rangle$. Let m be the smallest positive integer such that $mb \in \langle a \rangle$. Then m is even. Let H be the subgraph of $G(\Gamma; S)$ induced by $\cup\{ib + \langle a \rangle \mid i = 0, 1, \dots, m-1\}$.

If $m \geq 4$, let H_i be the subgraph of H induced by all the a -edges and b -edges of H in $(ib + \langle a \rangle) \cup ((i+1)b + \langle a \rangle)$, $i = 0, 2, 4, \dots, m-2$. Then $H_i \cong C_{2n+1} \times P_2$ and is 1-extendable. Hence if e_1 and e_2 are in different H_i 's, there is clearly a 1-factor of $G(\Gamma; S)$ containing e_1 and e_2 . Hence we may assume that e_2 is an edge in H_0 . For two independent edges $e_1 = 0a$ and e_2 in $C_{2n+1} \times P_2$, it is easy to find a 1-factor containing e_1 and e_2 except when $e_2 = (b+sa)(b+(s+1)a)$ for some $1 \leq s \leq 2n$ and $|\{0, 1\} \cap \{s, s+1\}| = 1$ (say $s = 1$). In this case, let $(m-1)b+ta$ be the vertex in H_{m-2} such that $mb+ta = 2a$. Let $M = \{e_1, e_2, (2a)((m-1)b+ta), b(2b)\} \cup \{(ka)(b+ka) \mid k \neq 0, 1, 2\}$. Then M can be extended to a 1-factor of $G(\Gamma; S)$, since the subgraphs induced by $(2b+\langle a \rangle) - \{2b\}$ and $((m-1)b+\langle a \rangle) - \{(m-1)b+ta\}$ are even paths

and the subgraph induced by $\cup\{ib + \langle a \rangle \mid i = 3, 4, \dots, m-2\}$ is isomorphic to $C_{2n+1} \times P_{2r}$, where $r = (m-4)/2$.

Finally, we need only to consider the case when $m = 2$. If $e_2 = (sa)((s+1)a)$, or $(b+sa)(b+(s+1)a)$ (for some s with $\{0, 1\} \cap \{s, s+1\} = \emptyset$), or $b(b+a)$, then as in the previous case, H has a 1-factor containing e_1 and e_2 . Hence, we may assume that $e_2 = (b+a)(b+2a)$. If $\langle a, b \rangle \neq \Gamma$, then there exists $c \in S - V(H)$. Let $K = V(H) \cup (c + V(H))$. Then K has the following 1-factor containing e_1 and e_2 :

$$M = \{e_1, e_2, c(c+a), (c+b+a)(c+b+2a), (2a)(2a+c), b(b+c)\} \\ \cup \{g(g+c) \mid g \in V(H) - \{0, a, 2a, b, a+b, 2a+b\}\}.$$

Hence M can be extended to a 1-factor of $G(\Gamma; S)$, as the subgraph of $G(\Gamma; S)$ induced by the set of all vertices of $G(\Gamma; S)$ not in K contains a spanning subgraph which is the disjoint union of a finite copies of H . Therefore it remains to consider the case that $\langle a, b \rangle = \Gamma$. As $G(\Gamma; S)$ is regular of degree at least 5, there exists $c \in S - \{a, -a, b, -b\}$. We may let $c = ta$ or $b + ta$ for some $1 < t \leq n$. If $c = ta$ with $t \geq 3$, let $M = \{e_1, e_2, (2a)((2+t)a), (ta)((t+1)a), b(b+ta), (b+(t+1)a)(b+(t+2)a)\} \cup \{(sa)(b+sa) \mid s \neq 0, 1, 2, t, t+1, t+2\}$. Then M is a 1-factor of containing e_1 and e_2 . If $c = 2a$, then let $M = \{e_1, e_2, (2a)(4a), (3a)(b+3a)\}$. Then M is an independent set containing e_1 and e_2 and the set of all vertices of $G(\Gamma; S)$ not in M can be decomposed into the disjoint union of even paths and so has a 1-factor M' . Hence $M \cup M'$ is a required 1-factor of $G(\Gamma; S)$. Now, we need only to consider the case that $c = b + ta$. As $c \neq b$ and $-b$, we have $t \neq 0$ and $2b \neq 2tna$, respectively. If $t \geq 3$, then let $M = \{e_1, e_2, c(a+c), (b+a-c)(b+2a-c)\} \cup \{x(x+c) \mid x \in \langle a \rangle - \{0, a, b+a-c, b+2a-c\}\}$. Then M is a 1-factor containing e_1 and e_2 . If $t = 1$, then let $M = \{e_1, e_2\} \cup \{x(x+c) \mid x \in \langle a \rangle - \{0, a\}\}$. Then M is a 1-factor containing e_1 and e_2 . Finally, we may assume that $t = 2$. Then $2b \neq 4na = (2n-1)a$. If $2b \neq 0$, then let $M = \{e_1, e_2, (2b+a)(2b+2a), (-b)(a-b)\} \cup \{y(y+b) \mid y \in (b+\langle a \rangle) - \{b+a, b+2a, -b, a-b\}\}$. Then M is a 1-factor containing e_1 and e_2 . On the other hand, if $2b = 0$, then let $M = \{e_1, e_2, (3a)(4a), (-c)(a-c)\} \cup \{y(y+c) \mid y \in (b+\langle a \rangle) - \{b+a, b+2a, -c, a-c\}\}$. Then M is a 1-factor containing e_1 and e_2 . $\square$

Chen, Liu and Yu [142] also obtained the classification of 2-extendable Cayley graphs on dihedral group. Actually, the only non-2-extendable Cayley graphs on dihedral group are precisely these five exceptions in Theorem 6.4.2.

Recently, the question of 2-extendability of Cayley graphs on dicyclic group was settled by Bai, Huan and Yu [64]. They also discussed 3-extendability of this class of graphs with low regularities. For Cayley graphs with regularity at most 5, they classify the 3-extendable graphs completely and conjectured that Cayley graphs on dicyclic group of regularity more than 5 are all 3-extendable.

Theorem 6.4.3 (Bai, Huan and Yu [64]). *Every Cayley graph on dicyclic group is 2-extendable.*

The problem of characterizing *all* 2-extendable Cayley graphs remains open, as well as the problem of classifying all 3-extendable Cayley graphs on Abelian, dihedral or dicyclic groups. However, Aldred, Holton and Lou (1993) obtained sufficient conditions in terms of cyclic-connectivity $c\lambda(G)$ for the k -extendability of symmetric graphs.

Theorem 6.4.4 (Aldred, Holton and Lou [15]). *Let G be an r -regular graph of even order satisfying $\lceil (r+1)/2 \rceil \leq k \leq r-1$ and $c\lambda(G) \geq (r-1)(2k-1)$. If*

- (a) G is vertex-transitive and $r \geq 4$; or
- (b) G is edge-transitive and $r \geq 3$,

then G is k -extendable.

We next consider the *generalized Petersen graph*, $PG(n, t)$ which has vertex-set $\{u_0, u_1, \dots, u_{n-1}, v_0, v_1, \dots, v_{n-1}\}$ and edge-set $\{u_i u_{i+1}, u_i v_i, v_i v_{i+t} \mid i = 0, 1, \dots, n-1\}$, where all subscripts are taken modulo n . Castagna and Prins [125] proved that all generalized Petersen graphs, except $GP(5, 2)$ (i.e., the usual Petersen graph), are 1-factorizable and thus 1-extendable. Since $PG(n, t)$ is a 3-regular graph with $2n$ vertices, it can not be 3-extendable. So it remains to determine which of them are 2-extendable. Schrag and Cammack [519] gave necessary and sufficient conditions for the 2-extendability of $GP(n, t)$, when $1 \leq t \leq 7$ and conjectured that $GP(n, t)$ is 2-extendable if and only if $n \neq 2t$ or $3t$ and $t \geq 3$. This conjecture was confirmed by Yu [615], which we state as a theorem below.

Theorem 6.4.5 (Schrag and Cammack [519]; Yu [615]). *Let $GP(n, t)$ be the generalized Petersen graph with $n \geq t$. Then*

- (a) $GP(n, 1)$ is 2-extendable if and only if n is even;
- (b) $GP(n, 2)$ is 2-extendable if and only if $n \neq 5, 6, 8$;
- (c) for $t \geq 3$, $GP(n, t)$ is 2-extendable if and only if $n \neq 2t$ or $3t$.

A graph G of order n is said to be *strongly regular* if it is k -regular and there exist two integers $\alpha, \beta \geq 0$ such that for every pair of vertices u, v ($u \neq v$), if u and v are adjacent then they have α common neighbors, while if they are not adjacent, they have β common neighbors. A strongly regular graph is often referred as an (n, k, α, β) -graph. This class of graphs possesses rich algebraic structures, and it has been studied widely by many researchers. Godsil and Royle's book [226] on algebraic graph theory has several chapters covered this topic.

Holton and Lou [249] showed that every strongly regular graph of even order with $\beta \geq 1$ is 1-extendable. Later, Lou and Zhu (1996) proved that all strongly regular graphs are 2-extendable except a few exceptions.

Theorem 6.4.6 (Lou and Zhu [397]). *A connected (n, k, α, β) -graph of even order with $k \geq 3$ is 2-extendable, except K_4 , the Petersen graph $GP(5, 2)$ or of type $(6, 4, 2, 4)$.*

The sufficient condition for 3-extendable strongly regular graphs was obtained by Holton and Lou (1992).

Theorem 6.4.7 (Holton and Lou [249]). *Let G be an (n, k, α, β) -graph of even order. If $k \geq 4$ and $k/2 \leq \beta \leq k - 1$, then G is 3-extendable, unless G is the $(6, 4, 2, 4)$ -graph.*

Chapter 7

Extremal k -Extendable Graphs and Generalizations

In this chapter, we study properties and structures of *extremal* k -extendable graphs and n -factor-critical graphs. By studying extremal graphs, either maximal or minimal, we are able to obtain effectively more understanding of the graphs, just as applying the extremal technique to other graph theory problems.

7.1 Maximal and minimal k -extendable graphs

In this section, we discuss the basic properties of families of extremal k -extendable graphs. In other words, we investigate the family of graphs which are k -extendable but no longer so after adding or deleting any new edge.

A graph G is called a *maximal k -extendable graph* (in short, *k -maximal*) if G is k -extendable but $G \cup e$ is not k -extendable for any edge $e \notin E(G)$.

Yu [614] proved that the only maximal 1-extendable graphs are K_n and $K_{n,n}$ (Saito (1991) also noted this fact). It was wondered if all k -extendable graphs are complete graphs and complete bipartite graphs. However, Györi (1992) pointed out that a Cartesian product of two complete graphs, $K_r \times K_s$, is a maximal k -extendable graph. So a natural question is to characterize all maximal k -extendable graphs.

For maximal bipartite graphs, the next theorem implies that $K_{n,n}$ are the only graphs.

Theorem 7.1.1 (Plummer [479]; Yu [614]). *If a connected bipartite graph $G = (X, Y)$ is k -extendable, then for any edge $e = xy \notin E(G)$ with $x \in X$ and $y \in Y$, $G \cup e$ is also k -extendable.*

Proof. Suppose the theorem does not hold. Then there exists a k -extendable bipartite graph $G = (X, Y)$ so that for some edge $e = x_0y_0$, where $x_0 \in X$, $y_0 \in Y$ and $x_0y_0 \notin E(G)$, $G \cup e$ is not k -extendable. Thus there exist k independent edges of $G \cup e$ which can not be extended to a 1-factor.

Since G is k -extendable, one of these edges must be e . Let the others be $e_1, e_2, \dots, e_{k-1}$, where $e_i = x_i y_i$ ($1 \leq i \leq k-1$), $x_i \in X$ and $y_i \in Y$. Let $X' = X - \{x_0, x_1, \dots, x_{k-1}\}$ and $Y' = Y - \{y_0, y_1, \dots, y_{k-1}\}$. Since (X', Y') has no 1-factors, by Hall's Theorem there exists a proper set $S \subseteq X'$ such that $|N_{Y'}(S)| < |S|$. But $|N_{Y' \cup \{y_0\}}(S)| = |N_{Y'}(S)| + 1$, thus we have $|N_{Y' \cup \{y_0\}}(S)| = |S|$. From Theorem 6.1.7 (c), the bipartite graph $(X' \cup \{x_0\}, Y' \cup \{y_0\}) = G - \{x_1, x_2, \dots, x_{k-1}, y_1, y_2, \dots, y_{k-1}\}$ is not 1-extendable, a contradiction to Observation 6.1.1. $\square$

For non-bipartite graphs, to determine all k -maximal graphs is a non-trivial problem. Some known examples of k -maximal graphs include K_n , $K_r \times K_s$. Three 2-maximal graphs are given in Fig. 7.1. The next theorem shows that the extendability of a graph can not be reduced too much after the addition of a new edge.

Theorem 7.1.2 (Yu [614]). *Let G be a connected non-bipartite graph of even order. If G is k -extendable ($k \geq 1$), then $G \cup e$ is $(k-1)$ -extendable for any $e \notin E(G)$.*

Proof. Assume that $G \cup e$ is not $(k-1)$ -extendable for some edge $e = xy \notin E(G)$. Since G is k -extendable, then there exist $k-1$ independent edges including e which can not be extended to a 1-factor. Let these edges be $e = xy$, $e_1 = x_1 y_1, \dots, e_{k-2} = x_{k-2} y_{k-2}$. Thus there are no 1-factors in the subgraph $G' = G - \{x, y, x_1, y_1, \dots, x_{k-2}, y_{k-2}\}$. By Tutte's 1-Factor Theorem, there exists a subset $S \subseteq V(G')$ such that $o(G' - S) \geq |S| + 1$ and a simple parity argument then yields $o(G' - S) = |S| + 2$.

Let $S' = S \cup \{x, y\}$. We show that S' is an independent set in $G'' = G - \{x_1, y_1, \dots, x_{k-2}, y_{k-2}\}$. Clearly, $o(G'' - S') = o(G' - S) = |S| + 2 = |S'|$. But since G'' is 2-extendable by Observation 6.1.1, it follows that G'' is 1-extendable (and also $o(G'' - S') = |S'|$). Theorem 6.1.1 (b) implies that S' is independent in G'' .

Next, we prove that each odd component of $G'' - S'$ is a singleton. If not, let O_1 be an odd component of $G'' - S'$ with $|O_1| \geq 3$. It is not hard to see that G'' is connected and moreover G'' is 3-connected and $|S'| \geq 3$, by Theorem 6.1.11 (b). From Lemma 6.1.1, there exist two independent edges f and g from O_1 to two vertices of S' , say u_1, u_2 . Let z_1 and z_2 be the end-vertices of f and g in O_1 and $S'' = S' - \{u_1, u_2\}$. Then we have

$$o((G'' - \{u_1, u_2, z_1, z_2\}) - S'') = o(G'' - S') = |S'| > |S''|.$$

That is, $G'' - \{u_1, u_2, z_1, z_2\}$ has no 1-factors, which contradicts to the fact that G'' is 2-extendable. So $|O_1| = 1$.

Since G'' is 2-extendable and connected, it follows that $G'' - S'$ has no even components as $o(G'' - S') = |S'|$. Thus all components of $G'' - S'$ are singletons, which implies that G'' is a bipartite graph.

Finally, we show that $G[V(G'') \cup \{x_1, y_1\}]$ is bipartite.

By Observation 6.1.1, $G^* = G[V(G'') \cup \{x_1, y_1\}] = G - \{x_2, y_2, \dots, x_{k-2}, y_{k-2}\}$ is 3-extendable, and so it is 4-connected. Let the bipartition of G'' be X'' and Y'' , where $|X''| = |Y''|$. Since G^* is 4-connected, each of x_1 and y_1 has degree at least 4. If G^* is not bipartite, either $N(\{x_1, y_1\}) \subseteq X''$ (or Y'') and G^* is not 2-extendable, or at least one of x_1 and y_1 has a neighbor in each of X'' and Y'' ; say x_1 does. But then y_1 has a neighbor in X'' or Y'' which is not a neighbor of x_1 and again G^* is not 2-extendable. So G^* is bipartite. Therefore, we can add each of $\{e_1, e_2, \dots, e_{k-2}\}$ to G'' one by one and conclude that $G[V(G'') \cup \{x_1, y_1, \dots, x_{k-2}, y_{k-2}\}] = G$ is bipartite. But this contradicts to the assumption. $\square$

It is important to point out that k -maximal graphs do not have the “hierarchy” property like what we have for k -extendable graphs, i.e., a k -maximal graph may not be necessarily $(k-1)$ -maximal. One such example is the first graph in Fig. 7.1. Actually, Porteous [497] noticed that “hierarchy” never happens to the class of k -maximal graphs. Applying Theorem 7.1.2, he proved the following.

Theorem 7.1.3 (Porteous [497]). *Suppose that $G \not\cong K_{n,n}$ or K_{2n} for any n . If G is k -maximal, then G is not $(k-1)$ -maximal.*

In their unpublished manuscript, Xu and Yu [597] obtained a characterization for k -maximal graphs. Denote a maximum matching in the subgraph induced by S in G by $M_G(S)$.

Theorem 7.1.4 (Xu and Yu [597]). *Let G be a graph with 1-factors. Then G is k -maximal ($k \geq 1$) if and only if*

- (a) *for any $S \subseteq V(G)$ with $|M_G(S)| \geq k$, $o(G-S) \leq |S| - 2k$, and*
- (b) *for any $uv \notin E(G)$, there exists $S \subset V(G)$ with $M_G(S) \cap \{u, v\} = \emptyset$, $\{u, v\} \subseteq S$ and $|M_G(S)| = k-1$ such that $o(G-S) = |S| - 2k + 2$.*

Proof. Since G is k -maximal, G is a k -extendable graph. By Theorem 6.1.2, (a) holds. For any $uv \notin E(G)$, $G' = G \cup \{uv\}$ is not k -extendable and thus there exists $S \subseteq V(G)$ with $|M_{G'}(S)| \geq k$ such that $o(G'-S) \geq |S| - 2k + 2$. We consider the following cases.

Case 1. $|S \cap \{u, v\}| \leq 1$.

Then $|M_G(S)| = |M_{G'}(S)| \geq k$. Since G is k -extendable, by Theorem 6.1.2, we have $o(G-S) \leq |S| - 2k$. Clearly, $o(G-S) \geq o(G'-S)$. Thus $|S| - 2k \geq o(G-S) \geq o(G'-S) \geq |S| - 2k + 2$, a contradiction.

Case 2. $|S \cap \{u, v\}| = 2$ and $|M_{G'}(S) \cap \{u, v\}| \leq 1$.

Clearly, $|M_G(S)| = |M_{G'}(S)| \geq k$. Since G is k -extendable, by Theorem 6.1.2 we have $o(G-S) \leq |S| - 2k$. Because $\{u, v\} \subseteq S$, then $o(G-S) = o(G'-S)$. Therefore, $|S| - 2k \geq o(G-S) = o(G'-S) \geq |S| - 2k + 2$, a contradiction.

Case 3. $|S \cap \{u, v\}| = 2$ and $uv \in M_{G'}(S)$.

Clearly, $|M_G(S)| \geq |M_{G'}(S)| - 1 \geq k - 1$. If $|M_G(S)| \geq k$, then that G is k -extendable implies $o(G - S) \leq |S| - 2k$. As $\{u, v\} \subseteq S$, so $o(G - S) = o(G' - S)$, which means that $o(G' - S) \leq |S| - 2k$, a contradiction. So $|M_G(S)| = k - 1$. Since G is k -extendable, by Theorem 6.1.11 (a), G is also $(k - 1)$ -extendable. Then $o(G - S) \leq |S| - 2k + 2$. Thus $|S| - 2k + 2 \geq o(G - S) = o(G' - S) \geq |S| - 2k + 2$. So we have $o(G - S) = |S| - 2k + 2$, where $\{u, v\} \subseteq S$, $M_G(S) \cap \{u, v\} = \emptyset$ and $|M_G(S)| = k - 1$.

Conversely, by Theorem 6.1.2, G is k -extendable. If G is not k -maximal, then there exists $uv \notin E(G)$ such that $G' = G \cup \{uv\}$ is still k -extendable. For any $S \subset V(G)$ with $M_G(S) \cap \{u, v\} = \emptyset$, $\{u, v\} \subseteq S$ and $|M_G(S)| = k - 1$, we have $|M_{G'}(S)| = k$. By Theorem 6.1.2, we have $o(G' - S) \leq |S| - 2k$, this contradicts to (b). So G is k -maximal. $\square$

The problem of determining the classes of k -maximal graphs for each k is still wide open, only a few special classes of k -maximal graphs (e.g., $k = 1, n - 1, n - 2$) are completely determined.

Theorem 7.1.5 (Ananchuen and Caccetta [38]). *Let G be a k -maximal graph of order $2n$. Then*

- (a) G is 1-maximal if and only if $G \cong K_{2n}$ or $K_{n,n}$;
- (b) G is $(n - 1)$ -maximal ($n \geq 2$) if and only if $G \cong K_{2n}$ or $K_{n,n}$;
- (c) G is $(n - 2)$ -maximal ($n \geq 5$) if and only if $G \cong K_{2n}$ or $K_{n,n}$;
- (d) For $n = 4$, besides K_8 and $K_{4,4}$, there are precisely three additional 2-maximal graphs (see graphs in Fig. 7.1).

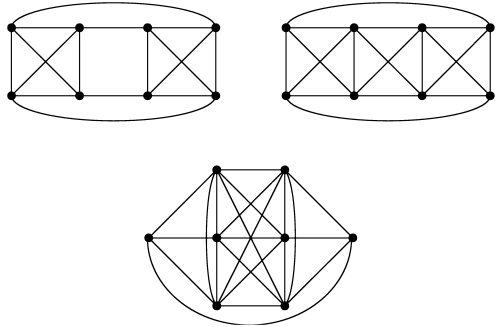


Fig. 7.1 Three maximal 2-extendable graphs.

From Theorem 7.1.1, we see that the maximal k -extendable bipartite graph must be complete. However, very little is known about the structures and properties of non-bipartite k -maximal graphs. But for the minimum degree of such graphs, we have the following result.

Theorem 7.1.6 (Ananchuen and Caccetta [38]). *If n, k are integers such that $1 \leq k \leq n - 1$ and $G \neq K_{2n}$ is k -maximal of order $2n$, then*

- (a) *If $n < 2k$, then $\delta(G) \leq n$;*
- (b) *If $n \geq 2k$, then $\delta(G) \leq n + 2 \lfloor \frac{k-1}{2} \rfloor$.*

To prove this theorem, we need two lemmas first.

Lemma 7.1.1 ([38]). *Let $G \neq K_{2n}$ be a k -maximal graph of order $2n$, $1 \leq k \leq n - 1$, and u and v any pair of non-adjacent vertices of G . Let M be a $(k - 1)$ -matching in $G - \{u, v\}$. Then the graph $G - \{u, v\} - V(M)$ has a matching of size at least $n - k - 1$.*

Lemma 7.1.2 ([38]). *Let G be a connected graph of order $2n$ with $\delta(G) \geq n - 1$ having a maximum matching M of size $n - 1$. Then $N_G(u) = N_G(v)$ for any two M -unsaturated vertices u and v of G . Furthermore, no two vertices of $N_G(u)$ are joined by an edge of M , and the vertices of $V(G) - N_G(u)$ form an independent set.*

Proof of Theorem 7.1.6. Let u and v be a pair of non-adjacent vertices of G and M a $(k - 1)$ -matching in $G - \{u, v\}$. Consider the graph $G' = G - \{u, v\} - V(M)$. Since G is k -maximal, G' has no 1-factors. Moreover, the subgraph $G[V(M) \cup \{u, v\}]$ has a maximum matching of size at most $k - 1$. We consider two cases.

Case 1. $n < 2k$.

Suppose that $\delta(G) \geq n + 1$. Let M' be a maximum matching in G' . By Lemma 7.1.1, $|M'| = n - k - 1$ (note that $|V(G')| = 2n - 2k$). Let x and y be M' -unsaturated vertices of G' . Clearly x and y are non-adjacent. Since $\delta(G) \geq n + 1$ and M' is a maximum matching of G' , there must be an edge e of M such that x and y are adjacent to distinct end-vertices of e , say a and b , respectively. Then $M' \cup \{xa, yb\}$ is a matching of size $n - k + 1 \leq k$ as $n < 2k$. Since G is k -extendable, $M' \cup \{xa, yb\}$ can be extended to a perfect matching of G , i.e., $G - (V(M') \cup \{x, a, y, b\})$ has a perfect matching. But $G - (V(M') \cup \{x, a, y, b\}) = G[(V(M) - \{a, b\}) \cup \{u, v\}]$ has a maximum matching of size at most $k - 2$, a contradiction. Hence $\delta(G) \leq n$.

Case 2. $n \geq 2k$.

Suppose that $\delta(G) \geq n + k$. Let $G_0 = G - \{u, v\}$. Then $|V(G_0)| = 2(n - 1)$ and $\delta(G_0) \geq \delta(G) - 2 \geq (n - 1) + (k - 1)$. By Theorem 6.3.21 (the case of $t = 2$), G_0 is $(k - 1)$ -extendable, which contradicts to the fact that G is k -maximal. Hence $\delta(G) \leq n + k - 1$ and we need only consider the case k even. For this case we prove that $\delta(G) \leq n + k - 2$.

Suppose that $\delta(G) = n + k - 1$. By the choice of G' , $\delta(G') \geq \delta(G) - 2k = n - k - 1$.

Claim. G' is connected.

If G' is not connected, then G' contains exactly two components as $|V(G')| = 2(n-k) \geq 2(\delta(G') + 1)$. In fact, G' consists of two disjoint K_{n-k} 's. Since G' has no 1-factors, $n-k$ and hence n must be odd.

As $\delta(G) = n + k - 1$, every vertex of G' must be adjacent, in G , to every vertex of $V(M) \cup \{u, v\}$. Let x and y be any two non-adjacent vertices of G' . Consider $G^* = G \cup xy$. We see that G' is connected by showing that G^* is k -extendable.

Suppose that G^* is not k -extendable. Then, since G is k -extendable, there exists a k -matching M^* , with $xy \in M^*$, that does not extend to a 1-factor in G^* . If $ab \in M^*$ and $a, b \notin V(G')$, then $(M^* - \{xy, ab\}) \cup \{xa, yb\}$ is a k -matching of G with vertex set $V(M^*)$. But then G can not be k -extendable, a contradiction. We get a similar contradiction with $ab \in M^*$ with $a \in V(G')$ and $b \notin V(G')$. Thus, we conclude that $V(M) \subseteq V(G')$. If $V(M) \neq V(G')$, then the graph $G'' = G - V(M) - V(M^*)$ consists of $2K_1 \vee (K_{2p} \cup K_{2q})$ for some p and q . Note that $V(2K_1) = \{u, v\}$. But G'' has a 1-factor implying that M^* is k -extendable. Hence $V(M^*) = V(G')$ and so $n-k = k$ implying that n is even, a contradiction. Therefore G^* is k -extendable, contradicting to the maximality of G . Hence G' is connected.

Since G' has no 1-factors, by Lemma 7.1.1, G' has a maximum matching M' of size $n - k - 1$. Let u' and v' be two M' -unsaturated vertices of G' . By Lemma 7.1.2, $N_{G'}(u') = N_{G'}(v')$. Let $N_{G'}(u') = \{x_1, x_2, \dots, x_{n-k-1}\}$. Lemma 7.1.2 implies that no two x_i 's are joined by an edge of M' and the set $V(G') - N_{G'}(u')$ is an independent set. Since $\delta(G) \geq n + k - 1$ and $G[V(M) \cup \{u, v\}]$ has a maximum matching of size at most $k - 1$, at least one of u or v , say u , is joined to a vertex, say w , of $N_{G'}(u')$. Consider the matching $M'' = M \cup \{uw\}$. The subgraph $G''' = G - V(M'')$ contains a set $S = \{v\} \cup (N_{G'}(u') - \{w\})$ such that $o(G''' - S) > |S|$. Hence G''' does not contain a 1-factor and so G is not k -extendable, a contradiction. This complete the proof. $\square$

The above theorem can be seen as a necessary condition for k -maximality, it provides a range for the feasible minimum degrees of k -maximal graphs. The bounds for $\delta(G)$ in Theorem 7.1.6 are sharp. For example, if $n < 2k$, the graph $K_{n,n}$ attains the bound; if $n = 2k$, Ananchuen and Caccetta [38] constructed a family of k -maximal graphs (e.g., $(K_k \cup K_k) \vee (K_k \cup K_k)$ when k is odd; $(K_{k-1} \cup K_{k+1}) \vee (K_{k-1} \cup K_{k+1})$ when k is even) attaining the bound.

Based on Theorem 6.1.15 of the last chapter, we can refine the range of minimum degree of k -maximal graphs in Theorem 7.1.6, which partially answers Open Problem 7.1.4.

Theorem 7.1.7 (Lou and Yu [394]). *Let G be a k -maximal graph of order $2n$. If $k \geq \frac{n+1}{2}$, then*

- (a) *if $k + 1 \leq \delta(G) \leq n$, then $G \cong K_{n,n}$ and hence $\delta(G) = n$;*
- (b) *if $2k + 1 \leq \delta(G) \leq 2n - 1$, then $G \cong K_{2n}$ and hence $\delta(G) = 2n - 1$.*

Proof. By Theorem 6.1.15, if $k + 1 \leq \delta \leq n$, then G is bipartite. Otherwise, $\delta(G) \geq \kappa(G) \geq 2k \geq n + 1$, a contradiction. When $n + 2 \leq 2k + 1$, $\delta(G) \neq 2k$ by Theorem 6.1.14. Hence $\delta(G) \geq 2k + 1 \geq n + 2$ and G is non-bipartite. By Theorem 7.1.1, (a) holds. By Theorem 7.1.6 (a), we see that (b) holds. $\square$

Ananchuen and Caccetta [38] also attempted to find sufficient conditions for k -maximal graphs. Among several conditions they found, the following one seems to be of strongly limited applicability, however it actually offers more practical insight than others.

Theorem 7.1.8 (Ananchuen and Caccetta [38]). *Suppose n, k and t are integers such that $1 \leq t \leq k \leq n - 1$. Let G be a k -extendable $(k + t)$ -regular graph of order $2n$ with diameter 2. Let w be any vertex of G and suppose u, v are two non-adjacent neighbors of w . If $G[N(w) - \{u, v\}]$ has exactly $t - 1$ independent edges, then G is k -maximal.*

Next, we investigate minimal k -extendable graphs, which can be seen as resulted graphs obtained by deleting edge one by one while maintaining the property of k -extendability. So let us first observe the impact of edge-deletion on extendability.

Theorem 7.1.9 (Plummer [480]). *If a graph G is connected and k -extendable ($k \geq 1$), then $G - e$ is $(k - 1)$ -extendable for any edge e of G .*

Proof. We proceed with induction on k .

For $k = 1$, since G is 1-extendable and 2-connected, there must be an edge f incident to the edge e and thus f can be extended to a 1-factor F_1 not containing e . Hence $G - e$ has a 1-factor.

Assume that the result holds for $k - 1$. We consider the case of k . Suppose that $G - e$ is not $(k - 1)$ -extendable. Then there exists a $(k - 1)$ -matching $M = \{e_1, e_2, \dots, e_{k-1}\}$, where $e_i = x_i y_i$, of $G - e$ which does not extend to a 1-factor of $G - e$. Since G is $(k - 1)$ -extendable, M can extend to a 1-factor F_2 and thus we may assume that every 1-factor of G which contains M also contains the edge e . In particular, $M \cup e$ is a k -matching of G .

Let $X = \{x_1, x_2, \dots, x_{k-1}\}$ and $Y = \{y_1, y_2, \dots, y_{k-1}\}$, and $e = xy$. If $N_{G-e}(x) \cup N_{G-e}(y) \not\subseteq X \cup Y$; say x is adjacent to vertex z where $z \notin X \cup Y$, then $M \cup \{xz\}$ is a k -matching in $G - e$ and can extend to a 1-factor F_3 of G not containing e . But this contradicts to the choice of M .

So we may assume that $N_{G-e}(x) \cup N_{G-e}(y) \subseteq X \cup Y$. Without loss of generality, assume $k \geq 3$, for if $k = 2$, either $\{x_1, y_1\}$ is a cutset of G , contradicting the fact that $\kappa(G) \geq 3$, or else $|V(G)| = 4 \leq 6 = 2k + 2 \leq |V(G)|$ which is also a contradiction.

Let $S = X \cup Y$. Since $|S \cup V(e)| = |S| + 2 = 2k$, and $|V(G)| \geq 2k + 2$, we must have that $T = G - (S \cup V(e)) \neq \emptyset$. However, S is a cutset of G separating x and y from T . By the $(k + 1)$ -connectedness of G , there are at least $k + 1$ vertices of S adjacent to vertices of T . If a vertex x_i is adjacent to a vertex

of T via an edge g . We claim that y_i does not lie in $N_{G-e}(x) \cup N_{G-e}(y)$. Otherwise, say $y_i \in N_{G-e}(x)$, then $M - e_i \cup g \cup xy_i$ is a k -matching not containing e in G and hence can extend to a 1-factor F_4 of G . But F_4 does not cover the vertex y , a contradiction.

Thus whenever a vertex x_i of S is adjacent to a vertex of T , then $y_i \notin N_{G-e}(x)$. But since $\kappa(G) \geq k+1$, at least $k+1$ vertices of S are adjacent to vertices of T and so at least $k+1$ vertices of S are not in $N_{G-e}(x) \cup N_{G-e}(y)$. Hence at most $2k-2-(k+1) = k-3$ vertices of S are in $N_{G-e}(x) \cup N_{G-e}(y)$. Hence $\kappa(G) \leq k-3$, a contradiction. $\square$

A graph G is called *minimal k -extendable* (in short, *k -minimal*) if G is k -extendable but $G - e$ is not k -extendable for any edge $e \in E(G)$.

From the definition of k -minimal graphs, we see that every k -extendable graph contains a k -minimal graphs. However, to characterize all k -minimal graphs for a fixed k is a non-trivial task. The examples of k -minimal graphs include: K_{2n} , $K_{n,n}$, $K_m \times K_2$ and $K_{n,n} - F$, where F is a 1-factor of the complete bipartite graph. Similar to the concept of k -maximality, k -minimality does not imply $(k-1)$ -minimality. In fact, Porteous (1995) proved the following.

Theorem 7.1.10 (Porteous [497]). *If G is k -minimal, then G is not $(k-1)$ -minimal.*

Moreover, k -maximality and k -minimality are two independent concepts, there is no implication from one to another. For example, a cycle C_{2n} is 1-minimal but not 1-maximal and in Fig. 7.2, the graph H_1 (e.g., $K_4 \times K_2$) is both 2-maximal and 2-minimal; H_2 is 2-maximal but not 2-minimal; H_3 is 2-minimal but not 2-maximal; H_4 is neither 2-maximal nor 2-minimal. It is interesting to note that there are graphs which are both k -maximal and k -minimal (e.g., K_{2n} , $K_{n,n}$ and H_1). This is a rather unusual phenomena.

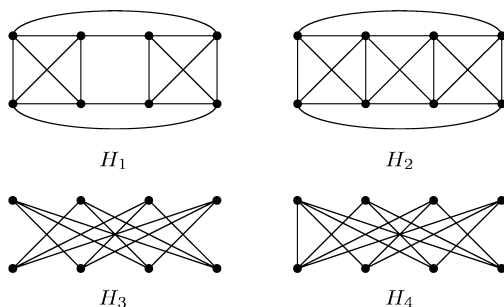


Fig. 7.2 Examples of 2-maximal and/or 2-minimal graphs.

Xu and Yu (2002) obtained a characterization for k -minimal graphs. Recall that $M_G(S)$ denotes a maximum matching in the subgraph induced by S in G .

Theorem 7.1.11 (Xu and Yu [597]). *Let G be a graph with 1-factors. Then G is k -minimal ($k \geq 1$) if and only if*

- (a) *for any $S \subseteq V(G)$ with $|M_G(S)| \geq k$, then $o(G - S) \leq |S| - 2k$, and*
- (b) *for any $uv \in E(G)$, let $G' = G - uv$, then there exists $S \subset V(G)$ with $S \cap \{u, v\} = \emptyset$, $|M_{G'}(S)| \geq k$ such that $o(G' - S) = |S| - 2k + 2$.*

Proof. Suppose that G is k -minimal. Then G is k -extendable. By Theorem 6.1.2, (a) holds. For any $uv \in E(G)$, $G' = G - uv$ is not k -extendable and thus there exists $S \subset V(G)$ with $|M_{G'}(S)| \geq k$ such that $o(G' - S) \geq |S| - 2k + 2$. Because $|M_G(S)| \geq |M_{G'}(S)| \geq k$ and G is k -extendable, by Theorem 6.1.2, $o(G - S) \leq |S| - 2k$. We consider the following two cases.

Case 1. $S \cap \{u, v\} \neq \emptyset$.

Clearly, $o(G - S) = o(G' - S)$. Then $|S| - 2k + 2 \leq o(G' - S) = o(G - S) \leq |S| - 2k$, a contradiction.

Case 2. $S \cap \{u, v\} = \emptyset$.

Clearly, $o(G - S) \geq o(G' - S) - 2$. Then $|S| - 2k \geq o(G - S) \geq o(G' - S) - 2 \geq |S| - 2k$, which implies $o(G' - S) = |S| - 2k + 2$ where $|M_{G'}(S)| \geq k$, and $S \cap \{u, v\} = \emptyset$.

Conversely, by (a) and Theorem 6.1.2, G is k -extendable. If G is not k -minimal, then there exists $uv \in E(G)$ such that $G' = G - uv$ is k -extendable. By Theorem 6.1.2, for any $S \subset V(G)$ with $|M_{G'}(S)| \geq k$, we have $o(G' - S) \leq |S| - 2k$, a contradiction to (b). $\square$

Even with the above characterization, it is still hard to derive general properties of k -minimal graphs. The next theorem is one of handful known results. The bound for the minimum degree in this theorem is sharp.

Theorem 7.1.12 (Ananchuen and Caccetta [41]). *If $1 \leq k \leq n - 1$ and $G \neq K_{2n}$ is a k -minimal graph of order $2n$, then $\delta(G) \leq n + k - 1$.*

Proof. If $k = n - 1$, then since $G \neq K_{2n}$, we have $\delta(G) \leq 2n - 2 = n + k - 1$ and we are done. Now we may assume $1 \leq k \leq n - 2$. Suppose, to the contrary, that $\delta(G) \geq n + k$. Let e be an edge of G . Since $G - e$ is not k -extendable, $G - e$ has a k -matching M such that M does not extend to a 1-factor of $G - e$. On the other hand, we have $\delta(G - V(M)) \geq n + k - 2k = n - k = (|V(G)| - |V(M)|)/2$ and $|V(G - V(M))| = 2(n - k) \geq 4$, since $k \leq n - 2$. Hence, by Dirac's Theorem, $G - V(M)$ has a Hamiltonian cycle of even order $2(n - k)$. Since every Hamiltonian cycle has two disjoint 1-factors, $G - V(M)$ has at least two disjoint matchings M_1 and M_2 . Clearly, either $e \notin M_1$ or $e \notin M_2$, since $M_1 \cap M_2 = \emptyset$. Without loss of generality, assume that $e \notin M_1$. But then $F = M_1 \cup M$ is a 1-factor of $G - e$ with $M \subseteq F$, contradicting to the assumption of M . Thus $\delta(G) \leq n + k - 1$. $\square$

When the graphs are restricted to be bipartite, the minimum degree can be actually determined.

Theorem 7.1.13 (Lou [385]). *If $G = (X, Y)$ is a k -minimal bipartite graph, then $\delta(G) = k + 1$. Furthermore, both X and Y contain at least $k + 1$ vertices of degree $k + 1$.*

Sketch of Proof. The main idea for proving this result is to introduce a new concept, a key set for a given edge. Let $G = (X, Y)$ be a minimal k -extendable bipartite graph and $e \in E(G)$. Define a *key set* for e to be a set S such that $S \subseteq X$ (or $S \subseteq Y$), $1 \leq |S| \leq |X| - k$ and $|N(S)| = |S| + k$ but $|N_{G-e}(S)| = |S| + k - 1$. The existence of such a set is assured by the k -extendability and the maximality.

The main steps of the proof include:

Step 1. Let $e = xy$ be an edge of G with $d(x) \geq n + 2$ and $d(y) \geq n + 2$ and X_1 be a minimum key set for e so that $X_1 \subseteq X$ and $x \in X_1$. If $f = xz$ is another edge with $d(z) \geq n + 2$ and $z \neq y$, then f has a minimum key set $T \subseteq Y$ such that $|T| < |X_1|$.

Step 2. Consider the subgraph H induced by the edges with both ends having degree at least $k + 2$. Use Step 1 to show that H is an acyclic graph.

Step 3. Based on Step 2, to show that either X or Y contains at least $k + 1$ vertices of degree $k + 1$.

Step 4. Enhance the result in Step 3 to show that both X and Y contain at least $k + 1$ vertices of degree $k + 1$. $\square$

For k -minimal bipartite graphs, Lou (1999) also determined the upper bounds for the number of vertices of degree $k + 1$ and the lower bounds for the number of vertices of degree $k + 2$.

Theorem 7.1.14 (Lou [385]). *If G is a k -minimal bipartite graph of order n , then*

- (a) *the number of vertices of degree $k + 1$ is at least $\lceil \frac{kn+2}{2k+1} \rceil$;*
- (b) *the number of vertices of degree $k + 2$ is at most $\lfloor \frac{kn+n-2}{2k+1} \rfloor$.*

From Theorem 7.1.13, we can easily derive the following.

Theorem 7.1.15 (Lou and Yu [394]). *Let G be a k -minimal graph of order $2n$ with $k \geq \frac{n}{2}$. If $k + 1 \leq \delta(G) \leq n$, then $\delta(G) = k + 1$. Furthermore, there are at least $2k + 2$ vertices of degree $k + 1$ in G .*

The problem of determining the classes of k -minimal graphs for each k is still on-going research topic, much work remains to be done. So far, only a few classes are completely clear.

Theorem 7.1.16 (Ananchuen and Caccetta [41; 40]). *Let n, k be integers such that $1 \leq k \leq n - 1$ and G be a graph of order $2n$. Then*

- (a) *K_{2n} or $K_{n,n}$ are k -minimal if and only if $k = n - 1$;*
- (b) *G is $(n - 1)$ -minimal ($n \geq 2$) if and only if $G \not\cong K_{2n}$ or $K_{n,n}$;*

- (c) if G is $(n-2)$ -minimal ($n \geq 3$), then $\delta(G) = n-1$, n , or $2n-3$. Moreover, if $n \geq 5$, then $\delta(G) \neq n$;
- (d) if $n \geq 5$ and G is $(n-2)$ -extendable, then G is $(n-2)$ -minimal if and only if
- (i) G is an $(n-1)$ -regular bipartite graph, or
 - (ii) G is a $(2n-3)$ -regular graphs, or
 - (iii) G contains one vertex of degree $2n-1$ and $2n-1$ vertices of degree $2n-3$, or
 - (iv) G contains $2n-2$ vertices of degree $2n-3$ and two vertices u and v of degree $2n-2$ such that $N(u) - v = N(v) - u$.

For smaller graphs, Ananchuen and Caccetta (see [40]) have exhausted all minimal graphs. When $|V(G)| = 6$, there are precisely four 1-minimal graphs. When $|V(G)| = 8$, there are precisely nine 2-minimal graphs.

More recently, Ananchuen, Caccetta and Ananchuen [45] introduced and studied maximal non- k -extendable graphs and maximal non- n -factor-critical graphs. They completely characterized these classes of extremal graphs.

A graph G is called *maximal non- k -extendable* if G is not k -extendable but $G \cup e$ is k -extendable for any edge $e \notin E(G)$. Similarly, G is called *maximal non- n -factor-critical* if G is not n -factor-critical but $G \cup e$ is n -factor-critical for any edge $e \notin E(G)$. As mentioned earlier, a $2k$ -factor-critical graph is obviously k -extendable but the converse need not be true. However, maximal non- k -extendable graphs and maximal non- $2k$ -factor-critical graphs have much more closed relation. In fact, for a connected graph G of order $2m$ with a perfect matching, G is maximal non- k -extendable if and only if G is maximal non- $2k$ -factor-critical for $1 \leq k \leq m-1$. With this equivalence, we only give the characterization of maximal non- n -factor-critical graphs.

Theorem 7.1.17 (Ananchuen, Caccetta and Ananchuen [45]). *Let G be a connected graph of order p and n a positive integer having the same parity with p . Then G is maximal non- n -factor-critical if and only if*

$$G \cong K_{n+s} \vee \bigcup_{i=1}^{s+2} K_{2t_i+1},$$

where s and t_i are non-negative integers with $0 \leq s \leq \frac{1}{2}(p-n)-1$ and $\sum_{i=1}^{s+2} t_i = \frac{1}{2}(p-n)-s-1$.

Actually maximal non- k -extendable graphs and maximal non- $2k$ -factor-critical graphs have some nice properties and close links with k' -extendable graphs and n' -factor-critical graphs. We summarize them in the next theorem.

Theorem 7.1.18 ([45]). *Let G be a maximal non- n -factor-critical graph of order p with $p \equiv n \pmod{2}$ and let H be a maximal non- k -extendable graph of order $2m$ with $1 \leq k \leq m-1$. Then*

- (a) G is $(n - 2)$ -factor-critical;
- (b) H is $(k - 1)$ -extendable;
- (c) $H \cup E'$ is k -extendable for any $E' \subseteq E(K_{2m}) - E(G)$ with $|E'| \geq 1$.

The study of extremal k -extendable graphs is a new direction in research of matching extension. The progress in exploration of structures and properties of these classes will be great help for understanding the k -extendable graphs in general. We include a few open problems here to stimulate interest.

Open Problem 7.1.1. Characterize k -maximal and k -minimal graphs for $2 \leq k \leq n - 3$, respectively.

Open Problem 7.1.2. Characterize graphs which are both k -maximal and k -minimal for $2 \leq k \leq n - 3$.

Open Problem 7.1.3. Find a family of infinite k -maximal/ k -minimal graphs (known classes: K_{2n} , $K_{n,n}$ and $K_r \times K_s$).

Open Problem 7.1.4. Suppose $1 \leq k \leq n - 1$ and $j \in [k + 1, n] \cup [2k + 1, 2n - 1]$. Which k -extendable graphs having minimum degree j are k -maximal?

7.2 Generalization of matching extension

There are several generalizations of matching extension. We give a brief review in this section.

Let G and H be graphs. Then a graph G is called (k, H) -extendable if G contains H as a subgraph and after deleting the vertices of any subgraph isomorphic to H the remaining subgraph, denoted by $G - H$, of G has a k -factor. This concept is a natural generalization of bicriticality (the case of $k = 1$ and $H = 2K_1$), n -extendability (the case of $k = 1$ and $H = nK_2$) and n -factor-criticality (the case of $k = 1$ and $H = nK_1$). Liu and Yu (1999) initiated the concept and obtained the following Tutte-type characterization.

Theorem 7.2.1 (Liu and Yu [361]). Let G be a graph with $|V(G)| \geq n + k + 1$ and H a fixed graph with $|V(H)| = n$. Then G is (k, H) -extendable if and only if for any two disjoint subsets S and T of $V(G)$ such that $G[S]$ contains H , then

$$\delta_G(S, T) = k|S| + \sum_{x \in T} d_G(x) - k|T| - e_G(S, T) - q_G(S, T) \geq kn.$$

Proof. Suppose that G is (k, H) -extendable. Let S and T be two disjoint subsets of $V(G)$ such that $G[S]$ contains H as a subgraph. Then $G' = G - V(H)$ has a k -factor. Let $S' = S - V(H)$. Then, it is easy to see $q_G(S, T) = q_{G'}(S', T)$. By Tutte's k -Factor Theorem, we have

$$\delta_{G'}(S', T) \geq 0.$$

Hence

$$\begin{aligned} \delta_G(S, T) &= \sum_{x \in T} d_{G-S}(x) - k|T| - q_G(S, T) + k|S| \\ &= \sum_{x \in T} d_{G'-S'}(x) - k|T| - q_{G'}(S', T) + k|S'| + k|V(H)| \quad (7.1) \\ &= \delta_{G'}(S', T) + k|V(H)| = \delta_{G'}(S', T) + kn \geq kn. \end{aligned}$$

Conversely, let $G' = G - V(H)$. We show that G' has a k -factor. For any two disjoint subsets S' and T' of $V(G')$, let $S = S' \cup V(H)$ and $T = T'$. Then by the assumption

$$\delta_G(S, T) \geq kn.$$

Thus

$$\begin{aligned} \delta_G(S, T) &= \sum_{x \in T} d_{G-S}(x) - k|T| - q_G(S, T) + k|S| \\ &= \sum_{x \in T} d_{G'-S'}(x) - k|T| - q_{G'}(S', T) + k|S'| + k|V(H)| \quad (7.2) \\ &= \delta_{G'}(S', T') + kn. \end{aligned}$$

Hence $\delta_{G'}(S', T') = \delta_G(S, T) - kn \geq 0$. By Tutte's k -Factor Theorem, G' has a k -factor or G is (k, H) -extendable. $\square$

This new concept has not been well studied yet. Only a few basic properties of (k, H) -extendable graphs, where $H = nK_1$, have been investigated.

Theorem 7.2.2 (Liu and Yu [361]). *Let G be a (k, nK_1) -extendable graph with $|V(G)| \geq n + k + 1$ and $n > 0$. Then*

- (a) *if k is even, then G is $(k, (n-1)K_1)$ -extendable;*
- (b) *if k is odd, then G is $(k, (n-2)K_1)$ -extendable.*

Theorem 7.2.3 (Liu and Yu [361]). *Let G be a connected graph. If G is $(k, 2K_1)$ -extendable, then $G - e$ has a k -factor for any given edge e .*

As seen in Section 3.1, toughness is a factor-friendly parameter. There are many results about factors in terms of toughness. So it is naturally to use toughness as a sufficient condition for the existence of (k, H) -extendable graphs. The following theorem was conjectured by Liu and Yu [361] and confirmed, independently, by Cai, Favaron and Li [106], and Enomoto [184].

Theorem 7.2.4 (Cai, Favaron and Li [106]; Enomoto [184]). *Let G be a graph and n a positive integer with $|V(G)| \geq n + 3$. If $t(G) \geq n \geq 3$, then G is $(2, (2n-2)K_1)$ -extendable.*

Next we discuss another generalization of matching extension introduced by Liu and Yu [362], which combines the concepts of matchings, extendability and factor-criticality.

Let d be a non-negative integer. A matching is called a *defect- d matching* if it covers exactly $|V(G)| - d$ vertices of G . Clearly, a defect-0 matching is a 1-factor.

Let G be a graph and let n, k and d be non-negative integers such that $|V(G)| \geq n + 2k + d + 2$ and $|V(G)| - n - d$ is even. If, when deleting any n vertices from G , the remaining subgraph of G contains k -matchings and each k -matching in the subgraph can be extended to a defect- d matching, then G is called an (n, k, d) -graph.

Clearly, a graph is a $(0, 0, d)$ -graph if and only if it has a defect- d matching and a graph is a $(0, 0, 0)$ -graph if and only if it has a 1-factor. Furthermore, $(0, k, 0)$ -graphs are precisely k -extendable graphs and $(n, 0, 0)$ -graphs are exactly n -factor-critical graphs.

A characterization of (n, k, d) -graphs is given below. Most known characterizations of different matching extensions in Section 6.1 can be deduced from this.

Theorem 7.2.5 (Liu and Yu [362]). *A graph G is an (n, k, d) -graph if and only if the following conditions hold:*

(a) *for any $S \subseteq V(G)$ with $|S| \geq n$, then*

$$o(G - S) \leq |S| - n + d;$$

(b) *for any $S \subseteq V(G)$ such that $|S| \geq n + 2k$ and $G[S]$ contains a k -matching, then*

$$o(G - S) \leq |S| - n - 2k + d.$$

Proof. First, we notice that Berge's Formula can be stated as that G has a defect- d matching if and only if for any $S \subseteq V(G)$, $o(G - S) \leq |S| + d$.

Suppose that G is an (n, k, d) -graph. Let S be a subset of $V(G)$ such that $|S| \geq n$. Let $U \subseteq S$ with $|U| = n$. By the definition of (n, k, d) -graphs, $G' = G - U$ has a defect- d matching. By Berge's Formula, for $S' = S - U$, we have

$$o(G' - S') \leq |S'| + d.$$

Thus

$$o(G - S) = o(G' - S') \leq |S'| + d = |S| - n + d,$$

that is, condition (a) holds.

Let $S \subseteq V(G)$ with $|S| \geq n + 2k$ such that $G[S]$ contains a k -matching M . Let $V(M) \subseteq S_1 \subseteq S$ and $|S_1| = n + 2k$. Set $S' = S - S_1$. Since G is an (n, k, d) -graph, $G' = G - S_1$ has a defect- d matching. By Berge's Formula, $o(G' - S') \leq |S'| + d$ or

$$o(G - S) = o(G' - S') \leq |S'| + d = |S| - n - 2k + d.$$

Thus condition (b) holds.

Conversely, suppose that both conditions (a) and (b) hold. For any $U \subseteq V(G)$ with $|U| = n$, we show that $G' = G - U$ has a k -matching. For any $S' \subseteq V(G')$, let $S = U \cup S'$. Then $|S| \geq n$. By condition (a),

$$o(G - S) \leq |S| - n + d.$$

Thus

$$o(G' - S') = o(G - S) \leq |S| - n + d = |S'| + d.$$

By Berge's Formula, G' has a defect- d matching. Since $n + 2k + d \leq |V(G)| - 2$, G' contains a k -matching. Next we show that any k -matching M of G' can be extended to a defect- d matching of G . Set $S_1 = U \cup V(M)$ and $G'' = G - S_1$. Then M can be extended to a defect- d matching of G' if and only if G'' has a defect- d matching. For any $S'' \subseteq V(G'')$, let $S_2 = S_1 \cup S''$. Then $|S_2| \geq n + 2k$ and $G[S_2]$ contains a k -matching. By condition (b),

$$o(G - S_2) \leq |S_2| - n - 2k + d,$$

or

$$o(G'' - S'') = o(G - S_2) \leq |S_2| - n - 2k + d = |S''| + d.$$

From Berge's Formula again, G'' has a defect- d matching. Therefore G is an (n, k, d) -graph. $\square$

The next theorem shows that (n, k, d) -graphs have the nice hierarchy relations.

Theorem 7.2.6 (Liu and Yu [362]). *Every (n, k, d) -graph is also an (n', k', d) -graph where $0 \leq n' < n$, $0 \leq k' < k$ and $n' \equiv n \pmod{2}$.*

One may also consider the trade-off between two parameters n and k in (n, k, d) -graphs.

Theorem 7.2.7 (Jin, Yan and Yu [263]). *For any $n > d \geq 0$ and $k \geq 2$, if G is an (n, k, d) -graph, then G is also an $(n + 2, k - 2, d)$ -graph.*

Proof. Suppose that G is not an $(n + 2, k - 2, d)$ -graph. Then there exist a vertex set S' of order $n + 2$ and a $(k - 2)$ -matching M' such that M' can not be extended to a defect- d matching of $G - S'$, i.e., $G - S' - V(M')$ has no defect- d matchings.

Claim. S' is an independent set in G .

If $e = uv$ is an edge in $G[S']$, then $e \cup M'$ can be extended to a defect- d matching of $G - (S' - u - v)$ since G is an $(n, k - 1, d)$ -graph, i.e., $G - S' - V(M')$ has a defect- d matching, a contradiction.

Let u, v be two vertices in S' and $G' = G \cup uv$. By Theorem 7.2.8, G' is an $(n, k-1, d)$ -graph. That is, $uv \cup M'$ can be extended to a defect- d matching M of $G - (S' - \{u, v\})$. Then M is also a defect- d matching of $G - S'$ which contains M' , a contradiction.

This completes the proof. $\square$

Note that when $d = 0$, Theorem 7.2.7 is still true without the condition $n > d$ (see [362]).

Jin, Yan and Yu (2007) investigated the effects of adding an edge, deleting an edge or adding a vertex to (n, k, d) -graphs. Their work generalized the results in [362] which considered these operations under the condition $d = 0$.

Theorem 7.2.8 (Jin, Yan and Yu [263]). *For any $n > d \geq 0$ and $k \geq 1$, if G is an (n, k, d) -graph, then $G \cup e$ is an $(n, k-1, d)$ -graph for any $e \notin E(G)$.*

Proof. For $k = 1$, since G is an $(n, 1, d)$ -graph, by Theorem 7.2.6, it is also an $(n, 0, d)$ -graph. Hence $G \cup e$ is an $(n, 0, d)$ -graph.

So assume that $k \geq 2$. If $G \cup e$ is not an $(n, k-1, d)$ -graph for some edge $e \notin E(G)$, then there exists an n -subset $S' \subseteq V(G)$ and a $(k-2)$ -matching $M' = \{x_1y_1, x_2y_2, \dots, x_{k-2}y_{k-2}\}$ such that $(k-1)$ -matching $e \cup M'$ can not be extended to a defect- d matching of $G - S'$. Let $e = xy$ and $S'' = V(M')$. By Berge's Formula, there exists a vertex set $S_1 \subseteq G - S' - S'' - x - y$ such that $o(G - S' - S'' - x - y - S_1) \geq |S_1| + d + 1$. Since G is an (n, k, d) -graph, according to Theorem 7.2.6, it is also an $(n, k-2, d)$ -graph. From Theorem 7.2.5 (b), $o(G - S' - S'' - x - y - S_1) \leq o(G - S' - S'' - S_1) + 2 \leq |S_1| + d + 2$. By a simple parity argument, we have $o(G - S' - S'' - x - y - S_1) = |S_1| + d + 2$. Let $S_2 = S_1 \cup \{x, y\}$. Then, $o(G - S' - S'' - S_2) = |S_2| + d$.

Claim 1. $S' \cup S_2$ is an independent set in G .

Suppose $e_1 = uv$ is an edge in $G[S' \cup S_2]$. Then $uv \cup M'$ is a $(k-1)$ -matching. Let $S = (S' \cup S_2 - u - v) \cup (S'' \cup \{u, v\})$ which is of order $|S_2| + n + 2(k-1) - 2$ and contains a $(k-1)$ -matching. Since G is an (n, k, d) -graph, according to Theorem 7.2.6, G is also an $(n, k-1, d)$ -graph. Then from Theorem 7.2.5 (b) and recall the fact that $|S_2| \geq 2$, we have

$$o(G - S' - S'' - S_2) = o(G - S) \leq |S| - n - 2(k-1) + d = |S_2| + d - 2,$$

a contradiction.

Let $H = G - S' - S'' - S_2$.

Claim 2. No even component of H is connected to $S' \cup S_2$.

Assume that there is an edge, say $e_2 = uv$, joining an even component C of H to $S_2 \cup S'$, where $u \in S' \cup S_2$ and $v \in V(C)$. Then $e_2 \cup M'$ is a $(k-1)$ -matching. Let $S = (S' \cup S_2 - u) \cup (S'' \cup \{u, v\})$ which is of order $n - 1 + |S_2| + 2(k-1)$ and contains a $(k-1)$ -matching. Since G is an (n, k, d) -graph, it is also an $(n, k-1, d)$ -graph. Hence Theorem 7.2.5 (b) implies that $o(G - S) \leq |S| - n - 2(k-1) + d = |S_2| - 1 + d$. However, since the total number

of odd components increases by at least one upon deleting v from the even component C , we have that $o(G-S) \geq o(G-S'-S''-S_2)+1 = |S_2|+d+1$, a contradiction.

Claim 3. For every odd component O of H , there do not exist two independent edges $e_3 = u_1v_1$ and $e_4 = u_2v_2$ joining O to $S' \cup S_2$, where $u_1, u_2 \in S' \cup S_2$ and $v_1, v_2 \in V(O)$.

Suppose, to the contrary, that e_3 and e_4 are two such edges. Then $e_3 \cup e_4 \cup M'$ is a k -matching. Let $S = (S' \cup S'' - u_1 - u_2) \cup (S'' \cup \{u_1, u_2, v_1, v_2\})$ which is of order $|S_2| + n - 2 + 2k$ and contains a k -matching. Since G is an (n, k, d) -graph, then according to Theorem 7.2.5 (b), we have

$$o(G-S) \leq |S| - n - 2k + d = |S_2| + n - 2 + 2k - n - 2k + d = |S_2| - 2 + d.$$

However, since the total number of odd components does not decrease by deleting v_1 and v_2 from the odd component O , we have $o(G-S) \geq o(G-S'-S''-S_2) = |S_2| + d$, a contradiction.

According to Claim 3, we conclude that for any odd component O of H , if it is connected to S_2 or S' in graph $G-S''$, then either $|N(V(O)) \cap (S' \cup S_2)| = 1$ or $|N(S' \cup S_2) \cap V(O)| = 1$.

Since G is an (n, k, d) -graph, $G-S''$ is an $(n, 2, d)$ -graph. Suppose that there are h odd components connected to neither S' nor S_2 , and t odd components $C_1, C_2, \dots, C_t$ with $|N(S' \cup S_2) \cap V(C_i)| = 1$, $1 \leq i \leq t$, and $p = |S_2| + d - h - t$ odd components $D_1, D_2, \dots, D_p$ with $|N(V(D_i)) \cap (S' \cup S_2)| = 1$, $1 \leq i \leq p$. Then $h + t + p = |S_2| + d$. Let $U = (\bigcup_{i=1}^p N(V(D_i))) \cap (S' \cup S_2) = \{u_1, u_2, \dots, u_q\}$. We consider the following three cases:

Case 1. $n \leq t$. Let $S_3 = (\bigcup_{i=1}^n V(C_i)) \cap N(S' \cup S_2)$. Then $|S_3| = n$. Now we consider the n -set S_3 and the $(k-2)$ -matching M' . From Claim 1, $S' \cup S_2$ is an independent set in $G-S''$. In $G-S''-S_3$, $S' \cup S_2$ must be matched by vertices of $|S_2| + d - h - n$ odd components from $C_{n+1}, C_{n+2}, \dots, C_t, D_1, D_2, \dots, D_p$ and any maximum matching of $G-S''-S_3$ must miss at least one vertex from each of h odd components which is connected to neither S' nor S'' . Altogether, a maximum matching of $G-S''-S_3$ misses at least

$$h + |S_2| + n - (|S_2| + d - h - n) = 2n + 2h - d \geq d + 2$$

vertices (recall that $n > d \geq 0$), which contradicts to the fact that $G-S''$ is an $(n, 2, d)$ -graph.

Case 2. $t < n \leq q + t$. Let $S_3 = (\bigcup_{i=1}^t V(C_i) \cap N(S' \cup S_2)) \cup \{u_1, u_2, \dots, u_{n-t}\}$. Now we consider the n -set S_3 and the $(k-2)$ -matching M' . Suppose that there are f odd components $D_{i_1}, D_{i_2}, \dots, D_{i_f}$ among $D_1, D_2, \dots, D_p$ which are connected to $\{u_1, u_2, \dots, u_{n-t}\}$ in $G-S''$. It is obvious that $f \geq n-t$. Note that each vertex of $(S' \cup S_2) - S_3$ can only be matched by vertices from $|S_2| + d - h - t - f$ odd components $\{D_1, D_2, \dots, D_p\} - \{D_{i_1}, D_{i_2}, \dots, D_{i_f}\}$

in $G - S'' - S_3$. Furthermore, any maximum matching of $G - S'' - S_3$ must miss at least one vertex from each D_{i_j} , $1 \leq j \leq f$, and at least one vertex from each of h odd components which is connected to neither S' nor S'' . Thus any maximum matching of $G - S'' - S_3$ must miss at least

$$\begin{aligned} f + h + |S_2| + n - (n - t) - (|S_2| + d - h - f - t) &= 2h + 2t + 2f - d \\ &\geq 2h + 2t + 2n - 2t - d \\ &\geq d + 2 \end{aligned}$$

vertices, which implies that $G - S''$ is not an $(n, 2, d)$ -graph, a contradiction again.

Case 3. $n > q + t$. Let $S_3 = (\bigcup_{i=1}^t V(C_i) \cap N(S' \cup S_2)) \cup U \cup S_4$, where $S_4 \subseteq S' \cup S_2 - U$ and $|S_4| = n - q - t$. Now we consider the n -set S_3 and the $(k - 2)$ -matching M' . Note that any maximum matching of $G - S'' - S_3$ must miss at least one vertex from each of the h odd components connected to neither S' nor S_2 and at least one vertex from $|S_2| + d - h - t$ odd components $D_1, D_2, \dots, D_p$. Furthermore, $|S_2| + n - (n - t)$ vertices of $S' \cup S_2 - S_3$ must be missed by any maximum matching of $G - S'' - S_3$. Thus any maximum matching of $G - S'' - S_3$ must miss at least

$$h + |S_2| + d - h - t + |S_2| + n - (n - t) = 2|S_2| + d \geq d + 4$$

vertices ($|S_2| \geq 2$), which implies that $G - S''$ is not an $(n, 2, d)$ -graph, a contradiction again.

This completes the proof. $\square$

For convenience, sometimes we also refer $(n, k, 0)$ -graphs as $(n, k, 0)$ -*extendable graphs*.

Theorem 7.2.9 (Liu and Yu [362]). *Let G be a connected $(n, k, 0)$ -extendable graph where $n \geq 2$, $k \geq 1$. Then for any edge e of G ,*

- (a) $G - e$ is $(n, k - 1, 0)$ -extendable;
- (b) $G - e$ is $(n - 2, k, 0)$ -extendable.

It was noted in [362] that Theorem 7.2.9 does not hold for (n, k, d) -graphs when $d > 0$. Jin *et al.* [263] provided structural theorems for $G - e$ to be an $(n, k - 1, d)$ -graph and an $(n - 2, k, d)$ -graph, respectively.

Theorem 7.2.10 (Jin, Yan and Yu [263]). *Let G be an (n, k, d) -graph with $k > 0$ and $n > d$. Then the join of G and a new vertex x , $G \vee x$, is an $(n + 1, k - 1, d)$ -graph.*

For the connectivity, we can prove the following result, which is a generalization of Theorems 6.1.11 (b) and 6.1.12 (b).

Theorem 7.2.11 (Liu and Yu [362]). *Every connected $(n, k, 0)$ -graph is $(n + k + 1)$ -connected, where $k \geq 1$.*

When $d > 1$, the connectivity of (n, k, d) -graphs can be quite small and it may not even relate to the parameter k . For example, let G be a k -extendable graph and G' be a graph obtained from G by adding a set S of d vertices so that $N_G(x) \neq \emptyset$ for any $x \in S$. Then it is not hard to verify that G' is a $(0, k, d)$ -graph. Note that the connectivity of G' can be any value in interval $[1, d]$ and so connectivity does not depend on k .

Many results about k -extendable graphs can be generalized to $(n, k, 0)$ -extendable graphs. We select one of such results and present it here, it is a generalization of Theorem 6.2.3 by Nishimura [461]. Given a vertex set S and a matching M , if $G - S - V(M)$ has a 1-factor, we say that G has an (S, M) -extension.

Theorem 7.2.12 (Liu and Yu [363]). *Let G be a graph with 1-factors and F an arbitrary 1-factor of G . If $G - V(e)$ is an $(n, k, 0)$ -extendable graph for each $e \in F$, then G is also an $(n, k, 0)$ -extendable graph.*

Proof. We may assume that $n > 0$ and $k > 0$.

We proceed to prove the theorem by contradiction. Suppose that there exists a 1-factor F of G such that $G - V(e)$ is $(n, k, 0)$ -extendable for any $e \in F$ but G is not $(n, k, 0)$ -extendable. Then there exist a k -matching M and a vertex set S of size n , where $V(M) \cap S = \emptyset$, such that $G - V(M) - S$ has no 1-factors. Let $G' = G - V(M) - S$. Applying Tutte's 1-Factor Theorem, there exists $S' \subseteq V(G')$ so that $o(G' - S') > |S'|$. By parity, $o(G' - S') \geq |S'| + 2$. Our aim is to find an edge $e \in F$ so that $G - V(e)$ is not $(n, k, 0)$ -extendable and thus leads to a contradiction.

At first, we show that 1-factor F can only match vertices from $V(M)$ to rest by the next claim.

Claim 1. For the given F , S and G' , we have

- (i) $F \cap E(S) = \emptyset$;
- (ii) $F \cap E(S') = \emptyset$;
- (iii) $F \cap E(S, S') = \emptyset$.

To see (i), if $e \in F \cap E(S)$, then $|S - V(e)| = n - 2$ and $G - V(e)$ is not $(n - 2, k, 0)$ -extendable. Thus, G is not $(n, k, 0)$ -extendable, a contradiction.

To see (ii), if $e \in F \cap E(S')$, then $G' - V(e)$ has no 1-factors or $G - V(e)$ is not $(n, k, 0)$ -extendable, a contradiction.

To see (iii), if $e \in F \cap E(S, S')$, where $e = ab$ and $a \in S$, $b \in S'$, choosing a vertex c from an odd component of $G' - S'$ and then $S - \{a\} \cup \{c\}$ and M can not be extended to a 1-factor as $o(G' - V(e) - S') > |S'| + 2 - 1$.

From (i)–(iii), it follows that a 1-factor F is in $E(S \cup V(M), G')$ or $E(S, V(M))$ or $E(G[V(M)])$.

Claim 2. G' has no even components.

Otherwise, let D be an even component and let $e = ab$ be an edge of F , where $a \in V(D)$.

If $b \in S$, choose $c \in V(D) - \{a\}$, then $T = S - \{b\}$ and M can not extended to a 1-factor in $G - \{a, b\}$ as $o((G' - V(e) - T - V(M)) - S') \geq |S'| + 2$, a contradiction.

If $b \in V(M)$, consider an alternating path of $M \cup F$ with end-vertex a . If another end-vertex c of this alternating path is in S , similar to the previous case, let $T = S - \{c\} \cup \{x\}$ (where $x \in V(D) - \{a\}$) and $M' = M - \{bc'\} \cup \{ab\}$, then $G - \{c, c'\}$ (where $cc' \in F$) has no (T, M') -extensions, a contradiction.

If c is in S' , it can be dealt similarly as in the previous paragraph.

If c is in a component (either odd or even), let $T = S$ and $M' = M - \{bc'\} \cup \{ab\}$, then $G - \{c, c'\}$ has no (T, M') -extensions as $G' - \{a, c\} - S'$ has at least $|S'| + 2$ odd components.

Claim 3. $S' = \emptyset$.

If $S' \neq \emptyset$, let $a \in S'$, then a is matched to a vertex b in the 1-factor F and b must be in $V(M)$. Consider an alternating path of $M \cup F$, say $abb' \dots dd'c$.

If $c \in S'$, let $T = S$ and $M' = M - \{bb', dd'\} \cup \{ab, b'd\}$, then $G - \{d', c\}$ has no (T, M') -extensions as $G' - \{a, c\}$ has no 1-factors.

If $c \in S$, let $T = S - \{c\} \cup \{x\}$ (where x is a vertex of some component) and $M' = M - \{bb', dd'\} \cup \{ab, b'd\}$, then $G - \{d', c\}$ has no (T, M') -extensions as $G' - \{a, c\} - (S' - \{a\})$ has $o(G' - S') - 1$ odd components, a contradiction.

If $c \in C$ (where C is any component), using the same argument we can see that $G' - \{a, c\} - (S' - \{a\})$ loses at most one odd component and obtain a contradiction.

Claim 4. $o(G' - S') = o(G') = 2$.

Suppose $o(G') > 2$ (i.e., $o(G') \geq 4$). If there exists an edge $e \in F$ and $e \in E(S, C_1)$, choose c from an odd component C_2 , let $T = S - \{b\} \cup \{c\}$ and $M' = M$, then $o(G' - \{a, c\}) \geq 2$ or $G - \{a, b\}$ has no (T, M') -extensions, a contradiction.

Otherwise, all vertices in $\cup_i C_i$ are matched into $V(M)$. Consider the alternating paths of $F \cup M$, there exists such a path starting with C_i and ending with C_j . Let $c_i x_1 y_1 x_2 y_2 \dots x_m y_m c_j$ be the alternating path, where $c_i \in C_i$, $c_j \in C_j$ and $c_i x_1, y_1 x_2, \dots, y_m c_j \in F$, $x_1 y_1, x_2 y_2, \dots, x_m y_m \in M$.

Let $T = S$ and $M' = M - \{x_1 y_1, \dots, x_m y_m\} \cup \{y_1 x_2, \dots, y_m c_j\}$. Then $G - \{c_i, x_1\}$ has no (T, M') -extensions as $o(G' - \{c_i, c_j\}) \geq 2$, a contradiction.

Claim 5. $F \cap E(S, V(M)) = \emptyset$.

Consider the alternating path $ab \dots c$ of $F \cup M$ with end-vertex a . If $c \in S$, let $T = S - \{a, c\}$ and $M' = M - \{bb'\} \cup \{cb'\}$, then $G - \{a, b\}$ does not have (T, M') -extension, that is, $G - \{a, b\}$ is not $(n - 2, k)$ -extendable, a contradiction. If $c \in C_1$ (where C_1 is an odd component) and $|C_1| \geq 3$, choose $d \in V(C_1) - \{c\}$ and let $T = S - \{a\} \cup \{d\}$ and $M' = M - \{bb'\} \cup \{b'd\}$. Then $G - \{a, b\}$ (where $ab \in F$) has no (T, M') -extensions as $o(G' - \{c, d\}) \geq 2$.

If $c \in C_1$ but $|C_1| = 1$, then we have $|C_2| \geq 3$ because G' has only two odd components, no even component and $|G'| \geq 4$. Suppose that $F \cap E(S, C_2) \neq \emptyset$. Let $e = gh \in F \cap E(S, C_2)$, where $g \in V(C_2)$ and $h \in S$. Choose $y \in$

$V(C_2) - \{g\}$ and set $T = S - \{h\} \cup \{y\}$ and $M' = M$, then $G - \{g, h\}$ has no (T, M') -extensions as $o(G' - \{g, y\}) \geq 2$, a contradiction.

So we may assume $F \cap E(S, C_2) = \emptyset$. In this case, all vertices of C_2 are matched to $V(M)$ in F . Considering $F \cup M$, there must be an alternating path with both end-vertices in $V(C_2)$ or an alternating path starting in $V(C_2)$ and ending in S . In either case, it yields a contradiction.

Now we are ready to conclude the proof.

Since $|S| \geq 1$ and $F \cap E(S, V(M)) = \emptyset$, there exists an edge $e = ab \in F$ from S to an odd component C_1 (where $a \in S$, $b \in V(C_1)$). If $|C_1| \geq 3$, let $c \in V(C_1) - \{b\}$ and set $T = S - \{a\} \cup \{c\}$ and $M' = M$, then $G - \{a, b\}$ has no (T, M') -extensions, a contradiction. If $|C_1| = 1$, then $|C_2| \geq 3$. Without loss of generality, we assume $F \cap E(S, C_2) = \emptyset$. Thus, all vertices of $V(C_2)$ are matched to $V(M)$ in F . Considering $F \cup M$, there exists an alternating path P with both of ends in C_2 or an alternating path P from C_2 to S .

Let $P = cx_1y_1d$, where $cx_1, y_1d \in F$ and $x_1y_1 \in M$. If $c, d \in V(C_2)$, let $T = S$ and $M' = M - \{x_1y_1\} \cup \{dy_1\}$, then $G - \{c, x_1\}$ has no (T, M') -extensions as $o(G' - \{c, d\}) \geq 2$. If $c \in V(C_2)$ and $d \in S$, let $T = S - \{d\} \cup \{g\}$ (where $g \in V(C_2) - \{e\}$) and $M' = M - \{x_1y_1\} \cup \{dy_1\}$, then $G - \{c, x_1\}$ has no (T, M') -extensions, a contradiction.

The proof is completed. $\square$

The concept of k -extendability introduced by Plummer [478] was only defined for connected graphs of even order. Naturally, we would like to extend this concept to graphs of odd order as well. One attempt is made by Yu [616] who introduced $k\frac{1}{2}$ -extendability; another attempt is from Wen [581] who introduced *defect k -extendable graphs*. But both attempts are special cases of the more general framework, (n, k, d) -graphs. They correspond to $(1, k, 0)$ -graphs and $(0, k, 1)$ -graphs, respectively. Even though these attempts are the special cases of (n, k, d) -graphs, but more properties and characterizations can be obtained in these special cases. For example, in her thesis, Wen [581] studied the connectivity of defect k -extendable graphs and determine the range of the possible values for the connectivity; using M -alternating paths to provide characterizations for defect k -extendable graphs and $(2n + 1)$ -factor-critical graphs. Furthermore, she conduct a through investigation of defect k -extendable *bipartite* graphs, including the research on minimum degree, minimal such graphs, hierarchy relations, etc.

Note that Li [344] modified the definition of (n, k, d) -graphs and introduced analogue version of (n, k, d) -graphs for bipartite graphs with $n > 0$. He present polynomial algorithms for $(0, k, d)$ -bipartite graphs and $(n, k, 0)$ -graphs, and showed a construction of $(0, k, d)$ -graphs from k -extendable graphs.

There is another generalization of matching extension introduced by Yuan [620]. A graph G is called *induced matching extendable*, or *IM-extendable* in short, if *every* induced matching of G can be extended to a perfect matching.

Not surprisingly, IM-extendable graphs are more difficult to study than k -extendable graphs. However, it is interesting to note that all IM-extendable graphs contain no large cycles. Denote $N^{(2)}(S) = \{u \in V(G) - S \mid d(u, S) = 2\}$. We need Turan's classical theorem as a lemma.

Lemma 7.2.1 (Turan's Theorem, Turan (1941)). *If G is a graph of order n , and contains no K_{m+1} 's, then $|E(G)| \leq |E(T_{m,n})|$ and the equality holds if and only if $G \cong T_{m,n}$, where $T_{m,n}$ is the complete m -partite graph of order n in which all parts are as equal in size as possible.*

Theorem 7.2.13 (Yuan [620]). *If G is a connected IM-extendable graph of order at least 4, then the girth $g(G) \leq 4$.*

Proof. At first, it is easy to see that every connected IM-extendable graph is 2-connected. Suppose, to the contrary, that G is a 2-connected graph with $g(G) \geq 5$. Then we have the following claims.

Claim 1. For $x, y \in V(G)$, $x \neq y$, then $|N(x) \cap N(y)| \leq 1$.

Otherwise, we have a cycle of length 4, a contradiction.

Let v be a vertex of G such that $d(v) = \delta(G) = k$. Let $N(v) = \{x_1, x_2, \dots, x_k\}$.

Claim 2. For any independent set $Y \subseteq N^{(2)}(v)$ of G , then $|Y| \leq k - 1$.

Otherwise, suppose $Y \subseteq N^{(2)}(v)$ is an independent set of G such that $|Y| = k$, and such that $|\{i \mid Y \cap N(x_i) \neq \emptyset, 1 \leq i \leq k\}|$ is as large as possible.

If there is some $x_i \in N(v)$ such that $Y \cap N(x_i) = \emptyset$, then there must be a vertex $x_j \in N(v)$ such that $|Y \cap N(x_j)| \geq 2$. Since $|Y - N(x_j)| \leq k - 2$, so by Claim 1 we have

$$|N(Y - N(x_j)) \cap N(x_i)| \leq |Y - N(x_j)| \leq k - 2.$$

Because of $|N(x_i) - \{v\}| \geq k - 1$, there exists $y_i \in N(x_i) - \{v\}$ such that $(Y \cup \{y_i\}) - N(x_j) \subseteq N^{(2)}(v)$ is independent in G . By Claim 1 again, $|N(y_i) \cap N(x_j)| \leq 1$, and so there exists a vertex $y_j \in Y \cap N(x_j)$ such that $Y' = (Y \cup \{y_i\}) - \{y_j\} \subseteq N^{(2)}(v)$ is an independent set of G with $|Y'| = k$. But this contradicts the choice of Y . Hence we must have $|Y \cap N(x_i)| = 1$, for $1 \leq i \leq k$.

Now suppose $Y \cap N(x_i) = \{y_i\}$, for $1 \leq i \leq k$. Then $M = \{x_i y_i \mid 1 \leq i \leq k\}$ is an induced matching of G , such that v is an isolated vertex in $G - V(M)$, a contradiction to the fact that G is IM-extendable. This completes the proof of Claim 2.

Note that, for $1 \leq i \leq k$, $d(x_i) \geq d(v) = k$, and $N(x_i) - \{v\} \subseteq N^{(2)}(v)$ is independent in G . Thus Claim 2 in fact implies that $d(v_i) = k$, for $1 \leq i \leq k$. Since G is connected, we deduce the following claim.

Claim 3. G is a k -regular graph.

Let $H = G[N^{(2)}(v)]$ be the subgraph of G induced by $N^{(2)}(v)$. Then $|V(H)| = k(k - 1)$ and thus Claim 3 implies the next claim.

Claim 4. $\Delta(H) \leq k - 1$.

Let $\overline{H}$ be the complement graph of H . Then $|V(\overline{H})| = |V(H)| = k(k-1)$, and $\delta(\overline{H}) = k(k-1) - 1 - \Delta(H) \geq k(k-2)$. Hence we have $|E(\overline{H})| \geq \frac{1}{2}k^2(k-1)(k-2) = |E(T_{k-1,k(k-1)})|$. By Claim 2, $\overline{H}$ has no k -cliques. Thus, by Lemma 7.2.1, $|E(\overline{H})| \leq |E(T_{k-1,k(k-1)})|$ and so we have $|E(\overline{H})| = |E(T_{k-1,k(k-1)})|$. Now by Lemma 7.2.1 again, we deduce that $\overline{H} \cong T_{k-1,k(k-1)} = T_{k,k,\dots,k}$. This implies that each component of H is isomorphic to K_k . When $k = 2$, by Claim 3, $G \cong C_5$, and G is clearly not IM-extendable. When $k \geq 3$, $g(G) = g(H) = 3$, a contradiction to the assumption that $g(G) \geq 5$. This completes the proof. $\square$

Since determining IM-extendable graphs is a co-NP-complete problem (see Yang and Yuan [606]), it is unlikely to find a good characterization or to determine all such graphs. To study IM-extendabilities of special classes of graphs or to find sufficient conditions for IM-extendability become a logical next step of investigation. The families of graphs which have been investigated include bipartite graphs, regular graphs, claw-free graphs, out-planar graphs, K_4 -minor-free graphs and graphs with diameter 2. On the other hand, sufficient conditions, such as minimum degree, degree-sum and etc., for IM-extendable graphs were obtained (see Yuan [621] for more details and complete list of references).

Yuan (1998) also investigated the lower bound of $|E(G)|$ for IM-extendable graphs.

Theorem 7.2.14 (Yuan [620]). *If G is a connected IM-extendable graph of order $2n$, then $|E(G)| \geq 3n - 2$. Moreover, the equality holds if and only if $G \cong T \times K_2$, where T is a tree of order n .*

There were other attempts to research maximal (reps. minimal) IM-extendable graphs, or IM-extendabilities under operations of various graph products, but it seems that they are not very fruitful. However, the approach from the prospective of power graphs generates some interesting results.

Theorem 7.2.15 (see [621]). *Let G be a connected graph of even order and T a tree of even order. Then the following statements hold.*

- (a) *If G is locally connected, then G^2 is IM-extendable;*
- (b) *if G is 2-connected, then G^3 is IM-extendable;*
- (c) *if G has a 1-factor, then G^3 is IM-extendable;*
- (d) *T^2 is IM-extendable if and only if there are no vertices of even degree in T and the diameter of T is no more than 4.*

Finally, we present a conjecture on induced matching number $i\mu(G)$ as a conclusion of this section.

Conjecture 7.2.1 (see [621]). *Let G be a connected graph but not a complete graph. Then*

$$i\mu(G \cup e) \leq i\mu(G)$$

for any $e \notin E(G)$.

The above conjecture is equivalent to that a maximal (edge-wise) induced matching graph must be a complete graph.

7.3 Variations of graph extension

Besides (n, k, d) -graphs in the previous section, there were several other attempts to generalize the concept of matching extension, for example, $E(m, n)$ -extendability and $[k, n]$ -extendability. On the other hand, there were various attempts on finding cycles, especially Hamiltonian cycles and 2-factors, to include a given set of edges or vertices. We present samples of such results on cycle extensions.

A graph G is said to have property $E(m, n)$ or $E(m, n)$ -extendable if for every pair of disjoint matchings $M, N \subseteq E(G)$ of sizes m and n , respectively, there is a 1-factor F in G such that $M \subseteq F$ and $F \cap N = \emptyset$. We call F an (M, N) -extension. For example, the Petersen graph is $E(1, 1)$ -extendable but not $E(1, 2)$ -extendable or $E(2, 0)$ -extendable. This concept was introduced by Porteous and Aldred [498] as a generalization of k -extendability.

Although $E(m, n)$ -extendable graphs can be considered as a refinement for k -extendability, but it does not have the same nice hierarchy property like k -extendability and n -factor-criticality (i.e., Theorems 6.1.11 and 6.1.12). That is, property $E(m, n)$ may not necessarily imply property $E(m, n - 1)$. As example, consider the graph $H = (n - 2m)K_2 \vee (2n - 6m)K_1$, where $m \geq 1$ and $n \geq 6m + 1$. It is not hard to check that H has property $E(m, n)$ but not $E(m, n - 1)$. Hence it becomes interesting to classify when the implication relation holds. In [498], Porteous and Aldred were able to classify the implications for all the m and n .

Let us start with the parameter m . Firstly, it is easy to see that an $E(m, n)$ -extendable graph must be $E(m, 0)$ -extendable.

Theorem 7.3.1 (Porteous and Aldred [498]). *If a graph G has property $E(m, n)$, then G has property $E(m - 1, n)$.*

Proof. Suppose that G of order p has property $E(m, n)$. Let M and N be disjoint matchings with $M, N \subseteq E(G)$ and $|M| = m - 1$, $|N| = n$. It is obvious that G has property $E(m, 0)$ and thus has property $E(m - 1, 0)$ by Theorem 6.1.11. Therefore, there exists a 1-factor M' in G that contain M . Since $p \geq 2(m + n + 1)$, we have $|M'| \geq m + n + 1$. Then $|M' \cap (E(G) - (M \cup N))| \geq 2$, so we may choose $e \in M' \cap (E(G) - (M \cup N))$. The edge e is independent of M

as it belongs to the same 1-factor M' , so with $M_1 = M \cup \{e\}$ then there exists an (M_1, N) -extension M'' in G . Clearly, M'' is also an (M, N) -extension and the result follows. $\square$

Next, we deal with the implications for the parameter n . The following theorem states that when n is relatively small with respect to m , the implication $E(m, n) \Rightarrow E(m, n-1)$ does hold.

Theorem 7.3.2 (Porteous and Aldred [498]). *If a graph G has property $E(m, n)$, then G has property $E(m, n-1)$, for $n \leq 2m+2$.*

Proof. Let G be a graph of order p with property $E(m, n)$ and $n \leq 2m+2$. Suppose, to the contrary, that G is not $E(m, n-1)$ -extendable. Let $M = \{e_1, e_2, \dots, e_m\}$ and $N = \{f_1, f_2, \dots, f_{n-1}\}$ be two disjoint matchings such that there does not exist an (M, N) -extension. Let $I = V(G) - V(G[M \cup N])$ be the set of vertices in G not incident with any edge in $M \cup N$. Since G has property $E(M, N)$, there exist no edges $e \in E(G) - (M \cup N)$ such that $V(e) \cap V(N) = \emptyset$, otherwise there exists an $(M, N \cup \{e\})$ -extension which is also an (M, N) -extension. Thus every edge uv with $u \in I$ has vertex v incident with an edge in N .

We consider two cases.

Case 1. $n-1 \leq m$.

As $p \geq 2(m+n+1) = 2m+(n-1)+4$, we have $|I| \geq 4$. Since G is $E(m, n)$ -extendable, so G is $E(m, 0)$ -extendable and thus $E(n-1, 0)$ -extendable as $n-1 \leq m$ and Theorem 6.1.11. Thus there exists a 1-factor M' in G containing N . However, this is impossible since every vertex in I is only adjacent to $V(N)$.

Case 2. $m+1 \leq n-1 \leq 2m+1$.

Set $q = (n-1) - m$, so $1 \leq q \leq m+1$. Let x be the number of edges in M independent of N and let c be the number of vertices incident with both an edge in M and an edge in N , so $c \geq m-x$. As $p \geq 2(m+n+1) = (2m-c)+2(n-1)+(4+c)$, then certainly $|I| \geq 4+c > m-x+2$. Since G has property $E(m, 0)$, so there exists a 1-factor M' in G containing the independent set $\{f_{q+1}, \dots, f_{q+m}\}$. Thus M' must match every vertex in I by independent edges to vertices incident with edges $f_1, \dots, f_q$, so in particular there exists $m-x+2$ independent edges, say $g_1, \dots, g_{m-x+2}$, between I and edges $f_1, \dots, f_q$. Now consider disjoint independent set $M_1, N_1 \subseteq E(G)$ given by $M_1 = \{f_1, \dots, f_m\}$ and $N_1 = \{f_{m+2}, \dots, f_{q+m}, e_1, \dots, e_x, g_1, \dots, g_{m-x+2}\}$. Now $|M_1| = m$ and $|N_1| = (q-1)+x+(m-x+2) = n$. Thus there exists an (M_1, N_1) -extension M'' in G . M'' must match every vertex $v \in V(G) - V(N)$ to a vertex $u \in V(\{f_{m+1}, \dots, f_{q+m}\})$, that is (at least) $2(m+2)$ vertices to $2q$ vertices, a contradiction as $q \leq m+1$. This completes the proof. $\square$

For $m \geq 1$ and $n \geq 6m+1$, we showed by an example that $E(m, n)$ -extendability does not imply $E(m, n-1)$ -extendability. Porteous and Aldred

[498] also constructed examples which do not preserve this implication relation for $m \geq 1$ and $n \geq 2m + 3$. Hence $E(m, n)$ property behaves unexpectedly for some graphs with $n \geq 2m + 3$. However, it was shown that the implication “ $E(m, n) \Rightarrow E(m, n - 1)$ ” holds when n is relatively large than m and for given m, n the order of graphs which does not have the implication “ $E(m, n) \Rightarrow E(m, n - 1)$ ” is restricted by a bound in terms of m, n . We summarize the classification of m, n with the implication of $E(m, n)$ in the next theorem (see Fig. 7.3).

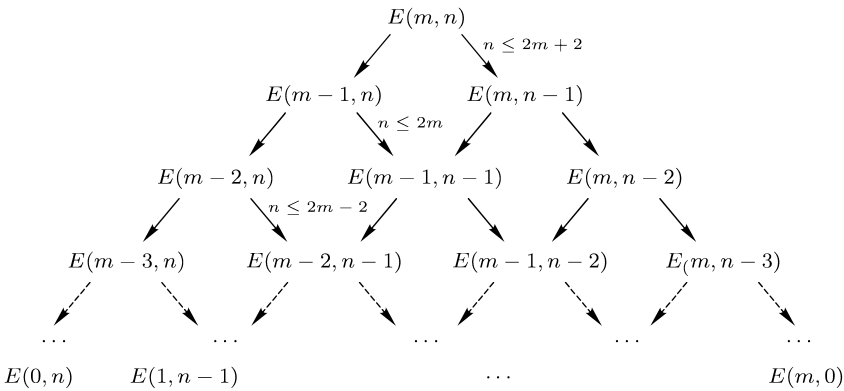


Fig. 7.3 Implication lattice for the general case $E(m, n)$.

Theorem 7.3.3. *Let G be a graph of order p with property $E(m, n)$. Then*

- (a) G has property $E(r, n)$, for any $r \leq m$;
- (b) G has property $E(m, r)$, for $n \leq 2m + 2$ and $r \leq n$;
- (c) G has property $E(m, r)$, for $n \geq m + 1$ and $r \leq m$;
- (d) G has property $E(m, r)$, if $p \geq 4(n - 1) - 2m$ and $r \leq n$.

On the other hand, if $n \geq 2m + 3$ and $m \geq 1$, there exists a graph H that has property $E(m, n)$ but not $E(m, n - 1)$.

Following along the same line as in the study of k -extendable graphs, we can discuss the sufficient conditions for property $E(m, n)$ using various graph parameters. We present a few results in this direction and we start with forbidden subgraph conditions.

Theorem 7.3.4 (Aldred and Plummer [22]). *Let m, n and r be non-negative integers with $m \geq 1$, $r \geq 3$ and let G be a $(2m + n + r - 2)$ -connected $K_{1,r}$ -free graph of even order at least $2m + 2n + 2$. Then G is $E(m, n)$ -extendable.*

The above theorem generalized the earlier results by Chen [138] (the case of $n = 0$) and Aldred *et al.* [18] (the case of $r = 3$).

Next we consider property $E(m, n)$ in planar graphs. Plummer [483] showed that no planar graph is $E(3, 0)$ -extendable and furthermore Aldred and Plummer [23] showed that no planar graph is $E(2, 1)$ -extendable. From Theorem 7.3.4, we see that planar graphs can only have properties $E(2, 0)$, $E(1, n)$ or $E(0, n)$ for certain n . With connectivity condition, Aldred and Plummer (2001) proved the following.

Theorem 7.3.5 (Aldred and Plummer [23]). *Let G be a planar graph of even order.*

- (a) *If G is 4-connected, then G is $E(1, 1)$ - and $E(0, 3)$ -extendable;*
- (b) *if G is 5-connected, then G is $E(1, 2)$ - and $E(0, 4)$ -extendable.*

Note that Plesník [477] proved that every r -regular $(r - 1)$ -edge-connected graph of even order has properties $E(0, r - 1)$ and $E(1, 0)$ (i.e., Theorem 1.4.3). Aldred and Plummer [24] proved that a 5-connected planar triangulation of even order is $E(0, 7)$ -extendable.

Another attempt is given by Liu and Yu [375], which is motivated in the study of extendability in product of graphs. For a graph G , let M be a k -matching of G and $U = \{u_1, u_2, \dots, u_n\}$ be a set of n distinct vertices of G such that $U \cap V(M) = \emptyset$. If there exists a 1-factor M^* of G such that $M \subseteq M^*$ and $u_i u_j \notin M^*$ for any pair $u_i, u_j \in U$, we call M^* an (M, U) -matching extension or a $[k, n]$ -extension. A graph G is called $[k, n]$ -extendable if it has 1-factors and there exists a matching extension of (M, U) for any given M and U as described above.

As an example, we consider the Petersen graph, which is $[2, 1]$ -extendable but not $[2, 2]$ -extendable. Clearly, $[k, 0]$ -extendability is just the usual k -extendability. It is not hard to see that a graph G is $[k, 0]$ -extendable if and only if it is $[k, 1]$ -extendable. Moreover, a $[k, n]$ -extendable graph is also $[k, p]$ -extendable for any $0 \leq p \leq n$.

In general, a basic relationship among $[k, n]$ -extendabilities is given by the next theorem.

Theorem 7.3.6 (Liu and Yu [375]). *If G is a connected $[k, n]$ -extendable graph, and $|V(G)| \geq 2k + 2n + 2$, then G is $[k - 1, n + 1]$ -extendable.*

Proof. Without loss of generality, we may assume that $n \geq 1$.

Case 1. $n > 1$. Arbitrarily choose a $(k - 1)$ -matching $M = \{v_1 w_1, v_2 w_2, \dots, v_{k-1} w_{k-1}\}$ and an $(n + 1)$ -set $U = \{u_1, u_2, \dots, u_{n+1}\}$, where $\{v_1, w_1, \dots, v_{k-1}, w_{k-1}\} \cap U = \emptyset$.

Firstly we claim that there exists an $(M, U - \{u_{n+1}\})$ -matching extension. If not, then for any $v \in V(G) - \{v_1, w_1, \dots, v_{k-1}, w_{k-1}, u_1, \dots, u_{n+1}\}$, $N_G(v) \subseteq \{v_i, w_i \mid i = 1, 2, \dots, k - 1\}$. Otherwise, if $vu \in E(G)$ and $u \notin \{v_i, w_i \mid i = 1, 2, \dots, k - 1\}$, then $(M \cup \{vu\}, U - \{u_{n+1}\})$ has a matching extension and so does $(M, U - \{u_{n+1}\})$, which contradicts the assumption.

But if $N_G(v) \subseteq \{v_i, w_i \mid i = 1, 2, \dots, k-1\}$, then v is an isolated vertex in $G - \{v_1, w_1, \dots, v_{k-1}, w_{k-1}\}$. Thus M can not be extended to a 1-factor of G , a contradiction.

Let M^* be an $(M, U - \{u_{n+1}\})$ -matching extension of G . If no $u_i u_{n+1} \in M^*$, then M^* is an (M, U) -matching extension. Otherwise, say $u_n u_{n+1} \in M^*$. Since $n > 1$, there exists a vertex $w \neq u_{n+1}$ so that $u_1 w \in M^*$. Let $M' = M \cup \{u_1 w\}$. Then $(M', U - \{u_1\})$ has a matching extension M^{**} . Clearly M^{**} is also an (M, U) -matching extension.

Case 2. $n = 1$. Let M^* be an $(M, U - \{u_1\})$ -matching extension. If $u_1 u_2 \notin M^*$, then M^* is an $(M, \{u_1, u_2\})$ -matching extension. So suppose that $u_1 u_2 \in M^*$ for any $(M, \{u_1\})$ -matching extension M^* . Let $W = \{v_i, w_i \mid i = 1, 2, \dots, k-1\} \cup \{u_1, u_2\}$. Then $N_G(u_1) \subseteq W$ and $N_G(u_2) \subseteq W$.

Claim 1. If one of $\{u_1, u_2\}$ is adjacent to v_j ($1 \leq j \leq k-1$), then the other must be adjacent to w_j .

Suppose that u_1 is adjacent to v_j for some j . Let $U = \{u_2\}$ and $M = \{v_1 w_1, v_2 w_2, \dots, v_{j-1} w_{j-1}, \dots, v_{k-1} w_{k-1}, v_j u_1\}$. Then (M, U) has a matching extension M^* . Thus $u_2 w_j \in M^*$ or u_2 is adjacent to w_j . A similar argument can be applied to u_2 .

Claim 2. Let $W' = N_G(u_1) \cup N_G(u_2)$. Then for any $v \in W'$, $N_G(v) \subseteq W' \subseteq W$.

Otherwise, there exists a vertex, say v_i , of W' such that $N_G(v_i) \not\subseteq W'$ and $v_i u_1 \in E(G)$. Let $u \in N_G(v_i) - W'$. Then $uv_i \in E(G)$. Thus $u_2 w_i \in E(G)$ by Claim 1. Denote the edges of M which are in W' by M' and let $M'' = M' - \{v_i w_i\} \cup \{uv_i, u_2 w_i\}$. Since $N_G(u_1) \subseteq W'$, u_1 is an isolated vertex of the graph obtained by deleting M'' with its end-vertices from G . Then M'' can not be extended to a 1-factor of G . But G is $[k, n]$ -extendable, thus G is also $[k, 0]$ -extendable. By Theorem 6.1.11, G is $[|M''|, 0]$ -extendable as $|M''| \leq k$, a contradiction. So Claim 2 is proved.

From Claim 2, we see that the induced subgraph $G[W']$ is a component of G . But $|V(G)| \geq 2k + 2n + 2 > |W'|$, so $G[W']$ is a proper subgraph of G . This contradicts to the fact that G is connected and thus there exists an $(M, \{u_1\})$ -matching extension M^* such that $u_1 u_2 \notin M^*$. Hence M^* is an $(M, \{u_1, u_2\})$ -matching extension. $\square$

This theorem is best possible. Consider the k -cube Q_k with 2^k vertices. From Theorem 6.3.30, Q_k is $(k-1)$ -extendable or $[k-1, 1]$ -extendable. By Theorem 7.3.6, Q_k is $[k-2, 2]$ -extendable. Since Q_k is a k -regular graph, clearly it is not $[k-2, 3]$ -extendable. Thus $[k, n]$ -extendability does not imply $[k-1, n+2]$ -extendability.

$[k, n]$ -extendability has many good properties and it behaves much like k -extendability, e.g., hierarchy relation.

Theorem 7.3.7 (Liu and Yu [375]). *If G is a $[k, n]$ -extendable graph of order at least $2k + 2n + 2$, then G is $[p, q]$ -extendable for $0 \leq p \leq k$, $0 \leq q \leq n$.*

Theorem 7.3.8 (Liu and Yu [375]). *Let G be a graph of even order. If $\delta(G) \geq |V(G)|/2 + k + n - 1$, then G is $[k, n]$ -extendable graph.*

The study of $[k, n]$ -extendability can be carried out paralleling to that of k -extendability, in particular in products of graphs. We include one here as a sample.

Theorem 7.3.9 (Liu and Yu [375]). *If G_1 is $[0, n]$ -extendable ($n \geq 1$) and let $\delta = \delta(G_1)$, then $G_1 \times G_2$ is $[0, n + \delta]$ -extendable.*

Now we turn back to the topics of Hamilton cycles and 2-factors. The reason prompting us to revisit these topics is that Hamilton cycles as well as 2-factors are well-studied and there are a lot of techniques and ideas towards these topics. The rest of this section intends to bring reader's attention to the existing results and approaches in the study of Hamilton cycles and 2-factors, hopefully one may absorb nutrition from these well-studied areas to inspire and to stimulate further research on factor extension theory.

The general setting of problems go like this: given a subgraph H , find a sufficient condition for the existence of an F -factor containing H . We present a group of assertions selected from recent publications. Often the techniques in adjusting Hamiltonian cycles or 2-factors are very involved and tedious, so we only list the results but not providing proofs and the interested readers can go to the sources directly for the proofs.

Let first consider Hamiltonian cycles with specified properties. The first such a result is due to Posa in 1963.

Theorem 7.3.10 (Posa (1963)). *Let G be a graph satisfying $\sigma_2(G) \geq n + k$. Then there exists a Hamiltonian cycle containing any given k edges.*

A graph is k -ordered Hamiltonian if for every ordered sequence of k vertices there is a Hamiltonian cycle that encounters the vertices of the sequence in the given order. This Hamiltonian property was introduced by Chartrand (see [444]). Clearly, every Hamiltonian graph is 3-ordered Hamiltonian.

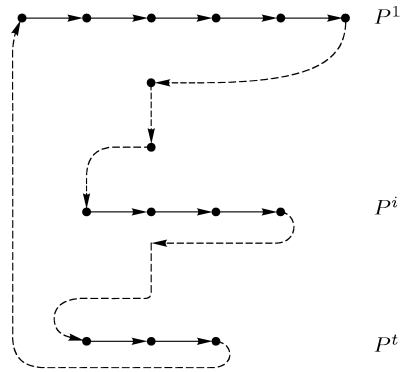
Theorem 7.3.11 (Ng and Schultz [444]). *Let G be a graph of order n and let k be an integer with $3 \leq k \leq n$. If $\sigma_2(G) \geq n + 2k - 6$, then G is k -ordered Hamiltonian.*

If restricting either the range of k to $3 \leq k \leq n/2$, or order of n to $n \geq 53k^2$, Theorem 7.3.11 can be improved. This is due to Faudree *et al.* (2000, 2003).

Theorem 7.3.12 (Faudree *et al.* [195; 201]). *Let $k \geq 3$ be an integer and let G be a graph of order n satisfying either $n \geq 53k^2$ or $3 \leq k \leq n/2$. If $\sigma_2(G) \geq n + (3k - 9)/2$, then G is k -ordered Hamiltonian.*

A natural variation of k -ordering would be to consider ordered sets of edges, ordered sets of vertices or even linear forests. We say L is a (k, t, s) -linear forest if L is a sequence $L = P^1, P^2, \dots, P^t$ ($1 \leq t \leq k$) of t disjoint paths, s of them being singletons such that $|V(L)| = k$. A graph G is (k, t, s) -ordered if for every (k, t, s) -linear forest L in G , there exists a cycle C in G that contains the paths of L in the designated order. Furthermore, if the paths of L are each oriented and C can be chosen to encounter the paths of L in the designated order and according to the designated orientation on each path, then we say G is *strongly* (k, t, s) -ordered (see an illustration in Fig. 7.4). If C is a Hamiltonian cycle, then we say G is (k, t, s) -ordered *Hamiltonian* and *strongly* (k, t, s) -ordered *Hamiltonian*, respectively. Note that saying G is (s, s, s) -ordered is the same as saying G is s -ordered. The following are two results in this direction.

Fig. 7.4 Strongly (k, t, s) -ordered graph where $L = P^1, P^2, \dots, P^t$ ($1 \leq t \leq k$) with s singletons such that $|V(L)| = k$.



Theorem 7.3.13 (Chen *et al.* [144]). For $1 \leq t \leq k$, if G is a (strongly) (k, t, s) -ordered graph on $n \geq k$ vertices with $\delta \geq \frac{n+k-t+s}{2}$, then G is (strongly) (k, t, s) -ordered Hamiltonian.

Theorem 7.3.14 (Chen *et al.* [144]). If $s = t = k \geq 3$ or $0 \leq s < t < k$, and G is a graph of order $n \geq \max\{178t + k, 8t^2 + k\}$ with

$$\sigma_2(G) \geq \begin{cases} n + k - 3 & \text{if } s = 0; \\ n + k + s - 4 & \text{if } 0 < 2s \leq t; \\ n + k + \frac{t-9}{2} & \text{if } 2s > t, \end{cases}$$

then G is strongly (k, t, s) -ordered.

Theorem 7.3.15 (Kouider [322]). Let G be a graph of order n and let k be an integer. If $\delta(G) \geq n/k$, then G can be covered by $k-1$ cycles, edges or vertices.

The above theorem clearly generalizes Dirac's Theorem for $k = 2$. The case $k = 3$ was also shown by Enomoto, Kaneko and Tuza (1988). Another

interesting approach is to extend a Hamiltonian cycle to a $[k, k+1]$ -factor under Ore-type condition. This is obtained by Cai, Li and Kano (1999), which involves more structural analysis than a Hamiltonian cycle.

Theorem 7.3.16 (Cai, Li and Kano [107]). *Let $k \geq 2$ be an integer and G a graph of order $n \geq 3$ with $\delta(G) \geq k$. Assume that $n \geq 8k - 16$ for even n and $n \geq 6k - 13$ for odd n . If $\sigma_2(G) \geq n$, then for any given Hamiltonian cycle C , G has a $[k, k+1]$ -factor containing C .*

A graph G is (k, m) -pancyclic, where $1 \leq k \leq m \leq n$, for any vertex set $S \subseteq V(G)$ with $|S| = k$, there exist cycles containing S and the length is between m and n . Faudree, Gould, Jacobson, Lesniak (2003) provided four degree conditions for the existence of (k, m) -pancyclic. For instance, $\delta(G) \geq (p+1)/2$ implies (k, m) -pancyclic.

Now we turn to the cycle extension problems, in particular 2-factors. We start with a recent result by Egawa *et al.* [173]. The proof of the theorem has a typical flavor in the area: firstly to show that appropriate degree conditions imply the existence of small cycles (length at most 4) which contain specified edges; secondly to extend these smaller cycles to a 2-factor containing the specified edges.

Theorem 7.3.17 (Egawa *et al.* [173]). *Let G be a graph of order p with $3k \leq p < 4k$ and $k \geq 2$. Let $e_1, e_2, \dots, e_k$ be independent edges of G . If $d_G(x) + d_G(y) \geq \lfloor \frac{p}{2} \rfloor + 4k - 2$ for nonadjacent vertices of $x \in X := \bigcup_{1 \leq i \leq k} V(e_i)$ and $y \in Y := V(G) - X$, or $\delta(G) \geq \lceil \frac{p+5k}{3} \rceil - 1$, then G has k disjoint cycles of length at most 4 containing k independent edges, respectively.*

For the vertex version of Theorem 7.3.17, Egawa *et al.* (2003) gave sufficient conditions in terms of minimum degree based on the range of the order $|V(G)|$.

Theorem 7.3.18 (Egawa *et al.* [170]). *Let G be a graph of order p . If for a positive integer k , one of the following conditions is satisfied,*

- (a) $p = 3k$, $\delta(G) \geq \frac{7k-2}{3}$, or
- (b) $3k+1 \leq p \leq 4k$, $\delta(G) \geq \frac{2p+k-3}{3}$, or
- (c) $4k \leq p \leq 6k-3$, $\delta(G) \geq 3k-1$, or
- (d) $p \geq 6k-3$, $\delta(G) \geq \frac{p}{2}$,

then for any set of k specified vertices $\{v_1, v_2, \dots, v_k\}$ there exists a 2-factor of G with k cycles C_i such that $v_i \in V(C_i)$ for $1 \leq i \leq k$.

For 2-factors with cycles of equal length, Yan and Liu (2004) proved the following result in bipartite graphs.

Theorem 7.3.19 (Yan and Liu [600]). *For a bipartite graph $G = (V_1, V_2; E)$ with $|V_1| = |V_2| = 2k$. If $d_G(x) + d_G(y) \geq 3k + 1$ for any pair of vertices of $x \in V_1$ and $y \in V_2$, then there is a $\{C_4\}$ -factor containing each edge of a given 1-factor.*

Holton and Plummer [252] introduced and studied the graphs having cycles containing a set of vertices and avoiding another set of vertices, as an extension of Dirac's Theorem. A graph G of order p is said to be $C(m, n)$ if $p \geq m + n$ and given arbitrary disjoint sets $M, N \subseteq V(G)$ of orders m, n respectively, there exists a cycle C in G that contains each vertex $u \in M$ and avoids each $v \in N$.

There are many other results involving 2-factors or $\{C_i\}$ -factors containing specified vertices or edges, the interested reader can refer to a survey paper by Faudree [194].

Chapter 8

Fractional Factors of Graphs

In this chapter, we discuss another generalization of factors, i.e., fractional factors, from the perspective of fractional edges instead of integer edges.

There is not much coverage of this topic in literature at this point, but the reader can refer to Scheinerman and Ullman's book [516] for more treatment on the topic of fractional matchings and Lovász and Plummer's book [412] (Chapter 7) for more discussions on fractional factors and its links with other branches of discrete mathematics, in particular to mathematical programming and polyhedron theory.

8.1 Fractional matchings

In Chapter 1, we introduced matchings in graphs and studied its properties and the sufficient conditions for the existence of matchings. In this section, we generalize matchings to fractional version, which is a natural extension from the discrete model (i.e., weighting edges with $\{0, 1\}$) to the continuous model (i.e., weighting edges from $[0, 1]$). This extension has also played an important role in study of integer programming and network flow theory, in particular matching polytopes initiated by Edmonds (1965). In this section, we follow closely the treatment in [516].

For a graph G , the *matching number* $\mu(G)$ of G is the size of a largest matching. Let h be a function that assigns each edge to a value in $[0, 1]$, i.e., $h : E(G) \rightarrow [0, 1]$. We call $h(e)$ the *weight* of e for an edge $e \in E(G)$. For a vertex $v \in V(G)$, let $d_G^h(v) = \sum_{e \in E_v} h(e)$, where $E_v = \{e \mid v \in V(e)\}$. We call $d_G^h(v)$ the *fractional degree* of vertex v . If $d_G^h(v) \leq 1$ holds for every vertex $v \in V(G)$, then h is called a *fractional matching function* of G . Let $E^h = \{e \mid e \in E(G) \text{ and } h(e) \neq 0\}$. If G_h is a spanning subgraph of G such that $E(G_h) = E^h$, then G_h is called a *fractional matching* of G with indicator function h , or simply a *fractional matching* of G (see Fig. 8.1). The *fractional matching number* $\mu_f(G)$ of a graph G is the supremum of

$\sum_{e \in E(G)} h(e)$ over all fractional matchings of G . The indicator function which achieves the supremum is called an *optimal* function.

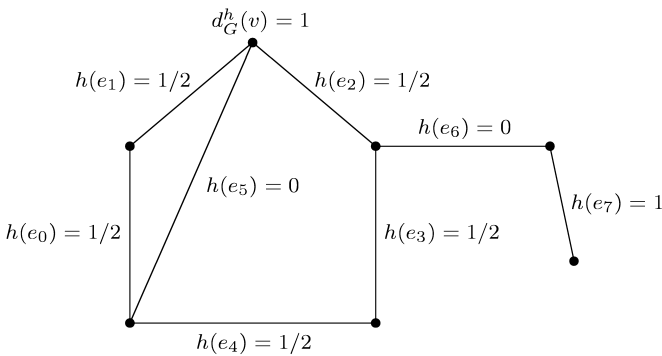


Fig. 8.1 An illustration of fractional matching function h .

Obviously, if $h(e) \in \{0, 1\}$ for every edge e , then a fractional matching G_h of G is simply a usual matching. In addition, it is clear to see that $\mu_f(G) \geq \mu(G)$ for all graphs G . As an example, consider C_5 (the cycle with 5 vertices), by assigning $\frac{1}{2}$ to every edge we obtain a fractional matching of C_5 . So we have $\mu_f(C_5) \geq \frac{5}{2}$. In fact, $\mu_f(C_5) = \frac{5}{2}$, by observing the following theorem.

Theorem 8.1.1 (see [516]). $\mu_f(G) \leq \frac{1}{2}|V(G)|$.

Proof. By the definition of fractional matchings, for any vertex $v \in V(G)$ and any fractional indicator function h of G , we have $d_G^h(v) \leq 1$. Summing the inequalities over all vertices v , it yields

$$\sum_{v \in V(G)} d_G^h(v) = \sum_{e \in E(G)} 2h(e) \leq |V(G)|.$$

Since h is an arbitrary indicator function, we have $\mu_f(G) \leq \frac{1}{2}|V(G)|$, and the theorem follows. $\square$

Similar to definitions of matchings and fractional matchings, we can define covering and fractional covering.

A *covering* of a graph G is a subset K of $V(G)$ such that every edge of G has at least one end in K . A covering K is a *minimum covering* if G has no covering K' with $|K'| < |K|$. The number of vertices in a minimum covering is called the *covering number* of G . A *fractional covering function* of a graph G is a function $p : V(G) \rightarrow [0, 1]$ satisfying $\sum_{v \in V(e)} p(v) \geq 1$ for every $e \in E(G)$. The *fractional covering number* is the infimum of $\sum_{v \in V(G)} p(v)$ taken over all fractional covering functions p of G .

By duality, the fractional covering number of a graph G is no more than the fractional matching number $\mu_f(G)$ of G . We now discuss the relationships between fractional matching number $\mu_f(G)$ and matching number $\mu(G)$ of a graph G , as well as those between matchings and coverings. The following classical result was proved by König-Egerváry (see [179]).

Theorem 8.1.2. *For a bipartite graph G , then the covering number equals to matching number, i.e., $\beta^*(G) = \mu(G)$.*

For the graph C_5 , we know that $\mu_f(C_5) > \mu(C_5)$. But for a bipartite graph, we have the following.

Theorem 8.1.3. *For any bipartite graph G , then $\mu_f(G) = \mu(G)$.*

Proof. Since G is bipartite, by Theorem 8.1.2, G has a covering K satisfying $|K| = \mu(G)$. Let h denote any fractional matching function of G , then

$$\sum_{e \in E(G)} h(e) \leq \sum_{v \in K} d_G^h(v) \leq \sum_{v \in K} 1 = |K|.$$

Hence $\mu_f(G) \leq |K| = \mu(G)$. Recall that $\mu_f(G) \geq \mu(G)$, so we have $\mu_f(G) = \mu(G)$. $\square$

Note that the above theorem can also be viewed as a direct consequence of Theorem 8.2.2. Next we present a formula for the fractional matching number $\mu_f(G)$ (see [516]), which is an analogue of Berge's matching formula.

Theorem 8.1.4. *For any graph G ,*

$$\mu_f(G) = \frac{1}{2}(|V(G)| - \max\{i(G - S) - |S|\}), \quad (8.1)$$

where the maximum is taken over all $S \subseteq V(G)$.

Let $D(G)$ be the set of all vertices in G which are missed by at least one maximum matching of G (note that $D(G)$ is the same notation used in Gallai-Edmonds Structure Theorem of Section 1.5). A necessary and sufficient condition for a graph G to have $\mu_f(G) = \mu(G)$ was obtained by Liu and Liu (2002).

Theorem 8.1.5 (Liu and Liu [378]). *Let G be a graph. Then $\mu_f(G) = \mu(G)$ if and only if $D(G)$ is an independent set of G .*

Proof. Suppose that $G[D(G)]$ is an independent set. Clearly, $\mu_f(G) \geq \mu(G)$. To see $\mu_f(G) \leq \mu(G)$, let $A(G) = N(D(G))$, by Theorems 8.1.4 and 1.5.4 (v), we have

$$\begin{aligned}
\mu_f(G) &= \frac{1}{2}(|V(G)| - \max\{|G - S| - |S|\}) \\
&\leq \frac{1}{2}(|V(G)| - (|D(G)| - |A(G)|)) \\
&= \frac{1}{2}(|V(G)| - \text{def}(G)) = \mu_G.
\end{aligned}$$

Hence $\mu_f(G) = \mu_G$.

Conversely, suppose $\mu_f(G) = \mu(G)$ but $G[D(G)]$ is not independent. Let G_1 be a non-singleton component of $G[D(G)]$. By Theorem 1.5.4 (i), then G_1 is factor-critical and $|G_1| \geq 3$. Let $v \in V(G_1)$. By the definition of $D(G)$, there exists a maximum matching M of G missing v . From Theorem 5.2.4, G_1 contains a nice (odd) cycle C_1 containing v . Let M_1 be a perfect matching of $G_1 - V(C_1)$ and $M' = M - E(G_1)$.

We construct a fractional matching h as follows:

$$h(e) = \begin{cases} 1 & e \in M' \cup M_1, \\ \frac{1}{2} & e \in E(C_1), \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned}
\mu_f(G) &\geq \sum_{e \in E(G)} h(e) \\
&= \frac{1}{2}(|V(G)| - \text{def}(G)) + 1 > \mu(G),
\end{aligned}$$

a contradiction. We complete the proof. $\square$

The upper bound $\frac{1}{2}|V(G)|$ of Theorem 8.1.1 is attained by the cycle C_n , which shows that the fractional matching number μ_f can take the value $\frac{n}{2}$ for any $n \geq 3$. To see that, we assign the number $\frac{1}{2}$ to each edge. The following theorem tells us that the value of $h(e) \in [0, 1]$ in a fractional matching can be restricted to $\{0, \frac{1}{2}, 1\}$.

Theorem 8.1.6 (Balinski [66]). *For any graph G , $2\mu_f(G)$ is an integer. Moreover, there exists a fractional matching function h for which*

$$\sum_{e \in E(G)} h(e) = \mu_f(G)$$

such that $h(e) \in \{0, \frac{1}{2}, 1\}$ for every edge $e \in E(G)$.

Proof. Among all the fractional matching functions h for which $\sum_{e \in E(G)} h(e) = \mu_f(G)$, choose one with the greatest number of edges e with $h(e) = 0$.

Denote this function by h^* and the corresponding fractional matching by H . We need several claims.

Claim 1. H has no even cycles.

Otherwise, let C be an even cycle of H and $m = \min_{e \in E(C)} h^*(e)$, where $h^*(e_0) = m$ for $e_0 \in E(C)$. Note that the maximum weight of $E(C)$ is at most $1 - m$. Let $g : E(G) \rightarrow \{-1, 0, 1\}$ to be the function that assigns 1 and -1 alternately to the edges of C with $g(e_0) = -1$ and assigns 0 to all the other edges of G . Then $h^* + mg$ is an optimal fractional matching function of G with at least one more edge, namely e_0 (see Fig. 8.2), assigned the weight 0, a contradiction to the choice of h^* . So H has no even cycles.

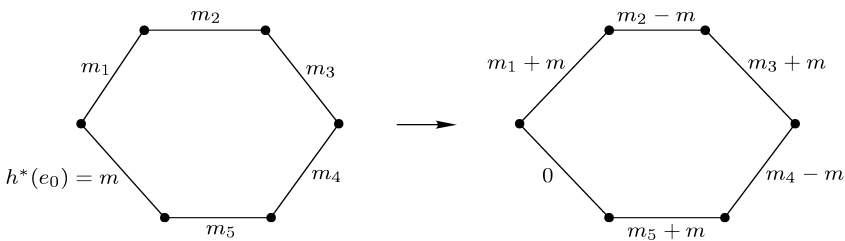


Fig. 8.2 Illustration of the change of h^* in Claim 1.

Claim 2. If H has a leave, then that edge is an entire component of H .

Otherwise, let $e = uv$ be a leave of H such that $d(u) = 1$ and $d(v) > 1$. Since e is not a component of H , we have $h(e) < 1$. Then increasing the weight of e to 1 and decreasing the weights of all other edges incident with v to 0, and this produces a new fractional matching function of G which attains the supremum and has more edges weighted 0, which contradicts the choice of h^* .

Claim 3. If H has an odd cycle C , then C must be an entire component of H .

Suppose that the claim does not hold. Let $v \in V(C)$ be a vertex such that $d_H(v) \geq 3$. Starting at v , we choose a longest path P in H which contains no edges of C . Note that the path P has no common vertices with C except for v . For, otherwise, it yields an even cycle. Moreover, by Claim 2, the degree of the other end vertex w of P is not 1. It is easy to see that all the vertices adjacent to w in H are on the path P . Thus H contains a subgraph K with two odd cycles C_1 and C_2 connected by a path P^* (possibly of length 0). Let $\varphi : E(H) \rightarrow \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$ be a function that assigns 0 to the edges not in K , ± 1 alternately to the path connecting the two cycles of K , and $\pm \frac{1}{2}$ alternately around the cycles of K so that $d_H^\varphi(v) = 0$ for each vertex $v \in V(H)$. For the function φ , it is clear to see that $\sum_{e \in E(H)} \varphi(e) = 0$.

Let

$$\varepsilon_1 = \min\{h^*(e) \mid e \in E(C_1) \cup E(C_2)\},$$

$$\varepsilon_2 = \min\{h^*(e) \mid e \in E(P^*)\},$$

and

$$\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}.$$

We consider two cases.

Case 1. $\varepsilon = \varepsilon_1$.

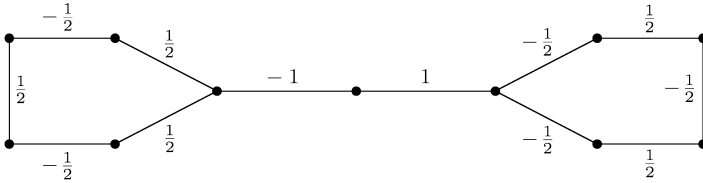
Set $m = 2\varepsilon_1$ or $-2\varepsilon_1$ such that $h' = h^* + m\varphi$ takes the value 0 on one of $E(C_1 \cup C_2)$ (see Fig. 8.3).

Case 2. $\varepsilon = \varepsilon_2$.

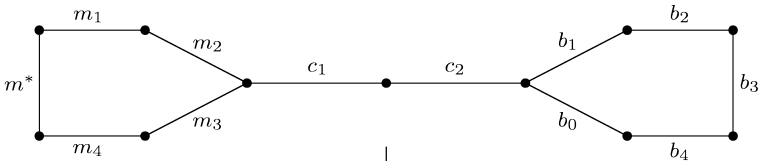
Set $m = \varepsilon_2$ or $-\varepsilon_2$ such that $h' = h^* + m\varphi$ takes the value 0 on one of $E(P^*)$.

Then $h' = h^* + m\varphi$ is a fractional matching function with more edges weighted 0 than h^* , contradicting to the choice of h^* .

φ :



h^* :



$h' = h^* + 2m^*\varphi$:

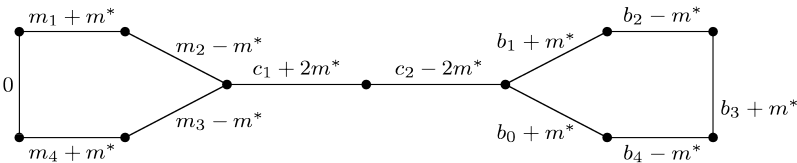


Fig. 8.3 Illustration of the change of h^* in Claim 3.

From the above three claims, we see that all the components of H are isomorphic to K_2 or odd cycles. If $e \in E(H)$ is a single edge in H , then let $h(e) = 1$; if e is in an odd cycle, then let $h(e) = \frac{1}{2}$; for the edges not in H , let $h(e) = 0$. It is easy to see that such a function h is a fractional matching function of G and h takes only the values $0, \frac{1}{2}, 1$, that is, $2\mu_f(G)$ is an integer. $\square$

The fractional matchings can also be viewed as solutions of mathematical programming problems. Let us associate each fractional matching with a vector $\mathbf{x} = (x_e \mid e \in E)$ satisfying two linear constraints: (i) $x_e \geq 0$ for all $e \in E$, (ii) $\sum_{e \sim v} x_e \leq 1$ for all $v \in V(G)$. Then a fractional matching is an extremal point of the polytope defined by (i) and (ii) if and only if $x_e \in \{0, \frac{1}{2}, 1\}$ for all $e \in E(G)$ and the edges e with $x_e = \frac{1}{2}$ form vertex disjoint odd cycles in G . Such fractional matchings constitute the set of basic feasible solutions.

The *support* of a fractional matching $\mathbf{x}$ is the subset $S(\mathbf{x})$ of E consisting of all those e for which $x_e \neq 0$. Then if $\mathbf{x}$ is a (basic) fractional matching, each component of the subgraph $(V, S(\mathbf{x}))$ is either an isolated vertex, a single edge or an odd cycle. The isolated vertices are unsaturated vertices by $\mathbf{x}$, the single edges are the integer parts of $\mathbf{x}$, and the odd components are the fractional parts of $\mathbf{x}$.

The fractional coverings are solutions of the feasible dual problem and so we can state dual version of Theorem 8.1.6 as follows.

Theorem 8.1.7 (see [516]). *For any graph G , there is a fractional covering function p for which*

$$\sum_{v \in V(G)} p(v) = \mu_f(G)$$

such that $p(v) \in \{0, \frac{1}{2}, 1\}$ for every vertex $v \in V(G)$.

Proof. Let $V(G) = \{v_1, v_2, \dots, v_n\}$ and $B(G)$ be the bipartite graph whose vertex set is $V(B(G)) = \{x_1, x_2, \dots, x_n\} \cup \{y_1, y_2, \dots, y_n\}$ with an edge $x_i y_j \in E(B(G))$ if and only if $v_i v_j \in E(G)$. In other words, $B(G)$ is the bipartite graph whose *bipartite* adjacency matrix is the adjacency matrix of G .

Given any fractional matching function $h : E(G) \rightarrow \{0, \frac{1}{2}, 1\}$, orient all the edges e of G for which $h(e) = \frac{1}{2}$ so that no two directed edges have the same head or tail. This can be easily archived, because the graph induced by the edges with $h(e) = \frac{1}{2}$ has maximum degree 2; simply orient the cycles cyclically and orient the paths linearly. Now we select edges of $B(G)$ as follows:

- (1) if $h(v_i v_j) = \frac{1}{2}$ and the edge $v_i v_j$ is oriented from v_i to v_j , then we choose the edge $x_i y_j$;
- (2) if $h(v_i v_j) = 1$, then we choose both edges $x_i y_j$ and $x_j y_i$.

The edges selected above form a matching of $B(G)$ and the cardinality of this matching is equal to $2 \sum_{e \in E(G)} h(e)$. Conversely, given any matching of

$B(G)$, we can define a fractional matching function $h : E(G) \rightarrow \{0, \frac{1}{2}, 1\}$ as follows: if both $x_i y_j$ and $x_j y_i$ are in the matching, let $h(v_i v_j) = 1$; if only one of these two edges is in the matching, let $h(v_i v_j) = \frac{1}{2}$; otherwise, let $h(v_i v_j) = 0$. Due to this correspondence, the fractional matching number of G is half of the matching number of $B(G)$.

Given an optimal fractional matching of G , we construct a maximum matching of $B(G)$ with cardinality $2 \sum_{e \in E(G)} h(e)$. By Theorem 8.1.2, there is a covering K of $B(G)$ of the same cardinality. From this covering, we can construct a fractional covering function $p : V(G) \rightarrow \{0, \frac{1}{2}, 1\}$ as follows: if both x_i and y_i are in the covering, let $p(v_i) = 1$; if only one is, let $p(v_i) = \frac{1}{2}$; otherwise, let $p(v_i) = 0$. Clearly, $\sum_{v \in V(G)} p(v) = \frac{1}{2}|K|$, so

$$\sum_{v \in V(G)} p(v) = \sum_{e \in E(G)} h(e) = \mu_f(G).$$

Hence, p is the fractional covering function which satisfies the conditions in the theorem. $\square$

Aharoni and Ziv (1990, 1991) had extended fractional matchings to that in infinite graphs and hypergraphs, in particular Theorems 8.1.6 and 8.1.7.

For a graph G , if it has a fractional matching function h such that $\sum_{e \in E(G)} h(e) = \frac{1}{2}|V(G)|$, in other words, if the fractional matching number $\mu_f(G)$ of G is equal to the upper bound $\frac{1}{2}|V(G)|$ in Theorem 8.1.1, then h is called a *fractional perfect matching function* of G . The fractional matching G_h with respect to h is called the *fractional perfect matching* of G . Obviously, if a fractional perfect matching function h takes only 0 and 1, then h is a (usual) perfect matching, or in other words, a 1-factor. In the following, we present necessary and sufficient conditions for a graph to have a fractional perfect matching.

Theorem 8.1.8 (see [516]). *Suppose h is a fractional matching function of a graph G . Then h is perfect if and only if $d_G^h(v) = 1$ for every vertex $v \in V(G)$.*

Proof. If $d_G^h(v) = 1$ for every vertex $v \in V(G)$, then

$$2 \sum_{e \in E(G)} h(e) = \sum_{v \in V(G)} d_G^h(v) = \sum_{v \in V(G)} 1 = |V(G)|.$$

Thus h is perfect.

Conversely, for, otherwise, if there exists a vertex $v \in V(G)$ such that $d_G^h(v) < 1$, then we have

$$2 \sum_{e \in E(G)} h(e) < |V(G)|,$$

contradicting to the definition of fractional perfect matching. $\square$

Theorem 8.1.9 (see [516]). *Let $G = (V, E)$ be a graph. Then the following statements are equivalent:*

- (a) G has a fractional perfect matching;
- (b) there is a partition $\{V_1, V_2, \dots, V_n\}$ of $V(G)$ such that, for each i ($1 \leq i \leq n$), the induced subgraph $G[V_i]$ is either a K_2 or a cycle;
- (c) there is a partition $\{V_1, V_2, \dots, V_n\}$ of $V(G)$ such that, for each i ($1 \leq i \leq n$), the graph $G[V_i]$ is either a K_2 or an odd cycle.

Proof. We follow the process of (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b). If G has a fractional perfect matching, then by Theorem 8.1.6, it has a fractional perfect matching function h with $h(e) \in \{0, \frac{1}{2}, 1\}$ for each $e \in E$. Then the support G_h of h is a spanning subgraph consisting of components either K_2 or a cycle. Clearly, the vertex sets of the components form a partition of $V(G)$ as desired.

(b) $\Rightarrow$ (c). If any set V_i , $i \in \{1, 2, \dots, n\}$, induces an even cycle, then it may be refined into sets of size two each of which induces a K_2 .

(c) $\Rightarrow$ (a). Now we construct a fractional perfect matching function h from the vertex partition. If a set V_i , $i \in \{1, 2, \dots, n\}$, induces a subgraph isomorphic to K_2 , let $h(e) = 1$ where e is the included edge. If a set induces an odd cycle, let $h(e) = \frac{1}{2}$ for every edge e on the cycle, and let $h(e) = 0$ for other edges e . The function h is clearly a fractional perfect matching function. $\square$

The most fundamental result in factor theory is Tutte's 1-Factor Theorem. In the following, we present a pretty analogous theorem for fractional perfect matchings, and referred it as Fractional Tutte's Theorem.

Theorem 8.1.10 (Fractional Tutte's Theorem). *A graph G has a fractional perfect matching if and only if*

$$i(G - S) \leq |S| \text{ for every set } S \subseteq V(G),$$

where $i(G - S)$ is the number of isolated vertices of $G - S$.

Proof. From Theorem 8.1.9, G has a fractional perfect matching if and only if G has a spanning subgraph with components K_2 's or odd cycles only, i.e., a $\{K_2, C_{2t+1} \mid t \geq 1\}$ -factor; then if and only if $i(G - S) \leq |S|$ for every set $S \subseteq V(G)$ from Theorem 4.2.5. $\square$

Tutte's 1-Factor Theorem gives us a necessary and sufficient condition for existence of a perfect matching in graphs, but it fails to determine the size

of maximum matching. The following result is an answer to this, which dues to Berge (1958) and is often referred as Berge's Formula.

Theorem 8.1.11 (Berge [76]). *For any graph G ,*

$$\mu(G) = \frac{1}{2} \left(|V(G)| - \max_{S \subseteq V(G)} \{o(G - S) - |S|\} \right),$$

where the maximum is taken over all $S \subseteq V(G)$.

The analogous result holds for the fractional matching number, and we call it Fractional Berge's Formula.

Theorem 8.1.12 (see [516]). *For any graph G ,*

$$\mu_f(G) = \frac{1}{2} \left(|V(G)| - \max_{S \subseteq V(G)} \{i(G - S) - |S|\} \right),$$

where the maximum is taken over all $S \subseteq V(G)$.

Proof. Suppose $i(G - S_0) - |S_0| = \max_{S \subseteq V(G)} \{i(G - S) - |S|\} = m$. The case $m = 0$ is exactly Fractional Tutte's Theorem, so we may assume that $m \geq 1$ in the following.

We first show that $\mu_f(G) \leq \frac{1}{2}(|V(G)| - m)$. Let $J \subseteq V(G)$ be the set of vertices that are neither in S_0 nor isolated in $G - S_0$. Let h be any fractional matching function of G , then

$$\sum_{e \in E(G)} h(e) \leq \sum_{v \in S_0} d_G^h(v) + \frac{1}{2} \sum_{v \in J} d_G^h(v),$$

since no edge connects an isolated vertex in $G - S_0$ to other isolated vertex in $G - S_0$ or to a vertex in J . Then

$$\begin{aligned} \sum_{e \in E(G)} h(e) &\leq |S_0| + \frac{1}{2}|J| = \frac{1}{2} (i(G - S_0) + |S_0| + |J| - (i(G - S_0) - |S_0|)) \\ &= \frac{1}{2} (|V(G)| - m). \end{aligned}$$

Conversely, to show $\mu_f(G) \geq \frac{1}{2}(|V(G)| - m)$, let $H = G \vee K_m$ be the join of G and K_m . We show that H has a fractional perfect matching. Choose $T \subseteq V(H)$. If $V(K_m) \not\subseteq T$, then either $i(H - T) = 0$ and so $|T| \geq i(H - T)$, or else $H - T$ is just a single vertex in K_m in which case $i(H - T) = 1 \leq |T|$. If $V(K_m) \subseteq T$, let $T' = T \cap V(G)$, then $i(H - T) = i(G - T')$, and so

$$|T| - i(H - T) = m + |T'| - i(G - T') \geq 0.$$

In all cases, we have $i(H - T) \leq |T|$. So by Theorem 8.1.10, H has a fractional perfect matching function h . The restriction of h to $E(G)$ is then a fractional matching function on G with

$$\begin{aligned} \mu_f(G) &\geq \sum_{e \in E(G)} h(e) = \sum_{e \in E(H)} h(e) - \sum_{e \in E(H) - E(G)} h(e) \\ &\geq \frac{|V(H)|}{2} - m = \frac{|V(G)| + m}{2} - m = \frac{|V(G)| - m}{2}, \end{aligned}$$

as required. $\square$

There are various sufficient conditions for existence of fractional perfect matchings. For example, Yu *et al.* (2006) gave a sufficient condition in terms of maximum degree for a graph to have a fractional perfect matching.

Theorem 8.1.13 (Yu *et al.* [610]). *Let G be a graph of order n . If $\max\{d(u), d(v)\} \geq \frac{n}{2}$ for any pair of nonadjacent vertices $u, v \in V(G)$, then G has a fractional perfect matching.*

8.2 Fractional (g, f) -factors

In this section we generalize (g, f) -factors to fractional (g, f) -factors in graphs, and present a necessary and sufficient condition for a graph having fractional (g, f) -factors. In the following, we always assume that g and f are two integer-valued functions defined on $V(G)$ such that $0 \leq g(v) \leq f(v)$ for every vertex $v \in V(G)$. Suppose φ is a real-valued function defined on a set S . We write $\varphi(S) = \sum_{x \in S} \varphi(x)$ for convenience and let $\varphi(\emptyset) = 0$.

Let h be a function defined on $E(G)$ such that $h(e) \in [0, 1]$ for every edge $e \in E(G)$. Recall that $E_v = \{e \mid e \sim v\}$ and $E^h = \{e \mid e \in E(G) \text{ and } h(e) \neq 0\}$. We call h a *fractional (g, f) -indicator function* of G if $g(v) \leq h(E_v) \leq f(v)$ for each $v \in V(G)$. If G_h is a spanning subgraph of G such that $E(G_h) = E^h$, then G_h is called a *fractional (g, f) -factor* of G with indicator function h , or simply a *fractional (g, f) -factor* of G . If no confusion arisen, we also call h a fractional (g, f) -factor for convenience. In particular, if $g(v) = f(v)$ for every $v \in V(G)$, then a fractional (g, f) -factor is also referred as a *fractional f -factor*. The *fractional $[a, b]$ -factors* and *fractional k -factors* can be defined similarly from their counterparts.

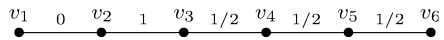
From the definitions above, it is easy to see that a (usual) (g, f) -factor is exactly a fractional (g, f) -factor with $h(e) \in \{0, 1\}$ for every edge $e \in E(G)$. So if G has a (g, f) -factor, then it must have a fractional (g, f) -factor, but the reverse may not be true. Fractional $[0, 1]$ -factors are precisely the fractional matchings discussed in the last section and fractional 1-factors are the fractional perfect matchings. For fractional k -factors, we have the following result.

Proposition 8.2.1. *If a graph G has a fractional k -factor, then it has a fractional m -factor, where $1 \leq m \leq k$.*

Proof. Suppose h is a fractional k -indicator function of G , then $\frac{m}{k}h$ is a fractional m -indicator function of G . Thus G has a fractional m -factor. $\square$

Let $F = G_h$ be a fractional $(0, f)$ -factor of G with indicator function h . A v -alternating path with respect to h is a path of G , $P(v_1, v) = v_1 v_2 \dots v_k (= v)$, such that $h(v_{2i-1} v_{2i}) < 1$ and $h(v_{2i} v_{2i+1}) > 0$ for each i . The path $P(v_1, v)$ is even (or odd) if k is odd (or even), denoted by $P_e(v_1, v)$ (or $P_o(v_1, v)$) (see Fig. 8.4). An *augmenting path* of G with respect to h is an odd v -alternating path $P_o(v_1, v)$ with $h(E_{v_1}) < g(v_1)$ and $h(E_v) < f(v)$, or an even v -alternating path $P_e(v_1, v)$ with $h(E_{v_1}) < g(v_1)$ and $h(E_v) > g(v)$. Let S and T be two disjoint subsets of $V(G)$. Recall that $E_G(S, T) = \{uv \mid uv \in E(G), u \in S, v \in T\}$, $e_G(S, T) = |E_G(S, T)|$, $e'_G(S, T) = h(E_G(S, T))$.

Fig. 8.4 An odd v -alternating path.



Next we present the main result of this section, which dues to Anstee [58] and is a necessary and sufficient condition for a graph to having a fractional (g, f) -factor. Recall that Lovász's (g, f) -Factor Theorem which involves considering all disjoint pairs of vertex subsets and so in some vague sense is a type of 2-parameter theorem. But Anstee's characterization involves only one set S since another set T is a function of S , namely

$$T = \{v \mid v \in V(G) - S, d_{G-S}(v) < g(v)\}.$$

Anstee (1985) proved Theorem 8.2.1 by an algorithm. The proof presented here is given by Liu and Zhang [369].

Theorem 8.2.1 (Anstee [58]). *A graph G has a fractional (g, f) -factor if and only if for any subset $S \subseteq V(G)$, we have*

$$g(T) - d_{G-S}(T) \leq f(S), \quad (8.2)$$

where $T = \{v \mid v \in V(G) - S, d_{G-S}(v) < g(v)\}$.

Proof. We first prove the necessity. If G has a fractional (g, f) -factor G_h with indicator function h , then for any $S \subseteq V(G)$, we have

$$h(E_G(S, V(G) - S)) \leq f(S).$$

It is easy to see that for $v \in T$,

$$h(E_G(S, \{v\})) \geq g(v) - d_{G-S}(v).$$

Thus we have

$$g(T) - d_{G-S}(T) \leq h(E_G(S, T)) \leq h(E_G(S, V(G) - S)) \leq f(S).$$

Now we assume that (8.2) holds and to show that G has a fractional (g, f) -factor. Otherwise, let $F = G_h$ be a fractional $(0, f)$ -factor which minimizes the quantity

$$\eta(F) = \sum_{v \in U} (g(v) - h(E_v)),$$

where $U = \{v \mid v \in V(G), h(E_v) < g(v)\}$.

Since F is not a fractional (g, f) -factor, we have $U \neq \emptyset$ and $\eta(F) > 0$. By the minimality of $\eta(F)$ we may assume that each edge e joining two vertices of U has $h(e) = 1$. Let

$S^* = \{v \mid v \notin U, \text{ there is an odd } v\text{-alternating path } P_o(v_1, v) \text{ of } G \text{ and } v_1 \in U\},$

$D = \{v \mid v \notin U, \text{ there exists an even } v\text{-alternating path } P_e(v_1, v) \text{ and } v_1 \in U\}.$

It follows from the minimality of $\eta(F)$ that $h(E_v) = f(v)$ for every $v \in S^*$. Otherwise, $h(E_v) < f(v)$. Let

$$\varepsilon_1 = \min_i \{1 - h(v_{2i-1}v_{2i})\},$$

$$\varepsilon_2 = \min_i \{h(v_{2i}v_{2i+1})\},$$

$$\varepsilon_3 = \min\{f(v_1) - h(E_{v_1}), f(v) - h(E_v)\},$$

and

$$\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}.$$

Let $h'(v_{2i-1}v_{2i}) = h(v_{2i-1}v_{2i}) + \varepsilon$, $h'(v_{2i}v_{2i+1}) = h(v_{2i}v_{2i+1}) - \varepsilon$, and $h'(e) = h(e)$ when $e \notin P_o(v_1, v)$ (see Fig. 8.5). Then $\eta(G_{h'}) < \eta(F)$, a contradiction to the selection of F .

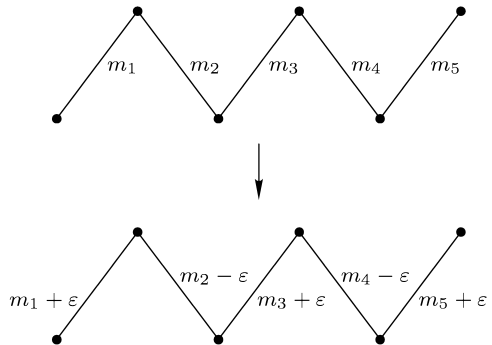


Fig. 8.5 Adjusting h to obtain h' .

Similarly, we have $h(E_v) = g(v)$ for $v \in D$. Otherwise, since $v \notin U$, $h(E_v) > g(v)$. Let $\varepsilon_4 = \min\{f(v_1) - h(E_{v_1}), h(E_v) - g(v)\}$ and $\varepsilon' = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_4\}$. Let $h''(v_{2i-1}v_{2i}) = h(v_{2i-1}v_{2i}) + \varepsilon'$, $h''(v_{2i}v_{2i+1}) = h(v_{2i}v_{2i+1}) - \varepsilon'$, and $h''(e) = h(e)$ for $e \notin P_e(v_1, v)$. Then $\eta(G_{h''}) < \eta(F)$, a contradiction again.

It follows from the above arguments that if F minimizes $\eta(F)$, then G does not admit an augmenting path with respect to h . Note that,

- (1) if $h(vw) > 0$, then $v \in S^*$ implies $w \in D \cup U$;
- (2) if $vw \in E(G)$ and $h(vw) < 1$, then $v \in D \cup U$ implies $w \in S^*$.

Now we prove that $S^* \cap D = \emptyset$. If $v \in S^* \cap D$, then there exists an odd alternating path $P_o(v_1, v) = v_1v_2 \dots v_{2k}(=v)$ and an even alternating path $P_e(w_1, v) = w_1w_2 \dots w_{2k+1}(=v)$, where $v_1, w_1 \in U$. If $P_o(v_1, v)$ and $P_e(w_1, v)$ have only one vertex in common, then $P_o(v_1, v) \cup P_e(w_1, v)$ is an augmenting path with respect to h , a contradiction. Otherwise, $P_o(v_1, v) \cup P_e(w_1, v)$ uses the same vertex or edge more than once, but no more than twice. Let

$$\varepsilon_5 = \min\{f(v_1) - h(E_{v_1}), f(w_1) - h(E_{w_1})\}$$

and

$$\varepsilon^* = \min\left\{\frac{1}{2}\varepsilon_1, \frac{1}{2}\varepsilon_2, \frac{1}{2}\varepsilon_5\right\}.$$

Let $h^*(v_{2i-1}v_{2i}) = h(v_{2i-1}v_{2i}) + \varepsilon^*$, $h^*(v_{2i}v_{2i+1}) = h(v_{2i}v_{2i+1}) - \varepsilon^*$, $h^*(w_{2i-1}w_{2i}) = h(w_{2i-1}w_{2i}) + \varepsilon^*$, $h^*(w_{2i}w_{2i+1}) = h(w_{2i}w_{2i+1}) - \varepsilon^*$, and $h'(e) = h(e)$ when $e \notin P_o(v_1, v) \cup P_e(w_1, v)$. Then G_{h^*} yields a smaller η , a contradiction.

Let $E_1 = \{e \mid e \in E(F) \text{ and } h(e) = 1\}$ and $F_1 = G[E_1]$. Then for $v \in D \cup U$, we have

$$d_{G-S^*}(v) = d_{F_1}(v) - e'_{F_1}(\{v\}, S^*) \leq h(E_v) - e'_{F_1}(\{v\}, S^*) \leq g(v).$$

Thus

$$\begin{aligned} g(T) - d_{G-S^*}(T) &\geq g(D \cup U) - d_{G-S^*}(D \cup U) \\ &= g(D \cup U) - d_{F_1}(D \cup U) + e'_{F_1}(D \cup U, S^*) \\ &\geq g(U) - d_{F_1}(U) + g(D) - d_{F_1}(D) + e'_{F_1}(D \cup U, S^*). \end{aligned} \quad (8.3)$$

Note the observation (1) above, we have

$$\begin{aligned} \sum_{v \in S^*} h(E_v) &= h(E_G(S^*, V(G) - S^*)) = f(S^*) = h(E_F(S^*, V(G) - S^*)) \\ &= h(E_F(S^*, D \cup U)) = e'_{F_1}(S^*, D \cup U). \end{aligned} \quad (8.4)$$

Set $S = \emptyset$ in (8.2), we obtain that $d_G(v) \geq g(v)$ for every $v \in V(G)$. Let $E_2 = \{e \mid e \in E(F) \text{ and } h(e) < 1\}$, $F_2 = G[E_2]$. Then by (8.3) and (8.4) we

have

$$\begin{aligned}
& g(T) - d_{G-S^*}(T) \\
& \geq g(D \cup U) - d_{F_1}(D \cup U) + e'_{F_1}(D \cup U, S^*) \\
& = g(D \cup U) - d_{F_1}(D \cup U) + e'_{F_2}(D \cup U, S^*) \\
& \quad - e'_{F_2}(D \cup U, S^*) + e'_{F_1}(D \cup U, S^*) \\
& \geq g(D \cup U) - d_{F_1}(D \cup U) + e'_F(D \cup U, S^*) - e'_{F_2}(D \cup U, S^*) \\
& = g(D) - d_{F_1}(D) - e'_{F_2}(D, S^*) + g(U) \\
& \quad - d_{F_1}(U) - e'_{F_2}(U, S^*) + f(S^*) \\
& = g(D) - \sum_{v \in D} h(E_v) + g(U) - \sum_{v \in U} h(E_v) + f(S^*) \\
& > f(S^*),
\end{aligned} \tag{8.5}$$

where we use the facts that $g(D) - \sum_{v \in D} h(E_v) = 0$, $g(U) - \sum_{v \in U} h(E_v) > 0$ and $e'_F(D \cup U, S^*) = f(S^*)$. But this contradicts to (8.2) and hence the proof is completed. $\square$

If $g(v) = f(v)$ for each vertex $v \in V(G)$, we obtain the following consequences from Theorem 8.2.1.

Corollary 8.2.1. *A graph G has a fractional f -factor if and only if*

$$f(T) - d_{G-S}(T) \leq f(S)$$

for any subset $S \subseteq V(G)$, where $T = \{v \mid v \in V(G) - S \text{ and } d_{G-S}(v) < f(v)\}$.

If $f(v) = k$ for all $v \in V(G)$ and some integer k , we obtain a characterization for fractional k -factors.

Corollary 8.2.2. *Let G be a graph. Then G has a fractional k -factor if and only if*

$$k|T| - d_{G-S}(T) \leq k|S|$$

for any $S \subseteq V(G)$, where $T = \{v \mid v \in V(G) - S \text{ and } d_{G-S}(v) < k\}$.

The following result is another form of Corollary 8.2.2, which is obtained by Zhang and Liu [627].

Corollary 8.2.3. *Let G be a graph. Then G has a fractional k -factor if and only if*

$$\sum_{j=0}^{k-1} (k-j)p_j(G-S) \leq k|S|$$

for any subset $S \subseteq V(G)$, where $p_j(G-S)$ denotes the number of vertices in $G-S$ with degree j .

Recall that fractional 1-factors are the same as fractional perfect matchings. So Corollary 8.2.3 implies Theorem 8.1.10, which is the case of $k = 1$.

An important feature of fractional factors is that they can be studied using network flow theory and duality theory in linear programming. Moreover, in some cases, fractional (g, f) -factors can be transformed into integral (g, f) -factors.

Let $G_{g=f}$ denote the subgraph of G induced by vertices x for which $g(x) = f(x)$. Anstee (1992) described four conditions under which fractional factors and (integral) factors are equivalent. We state two of them at the following theorem.

Theorem 8.2.2 (Anstee [60]). *Let G be a graph and $g, f : V(G) \rightarrow \mathbb{Z}$. If one of the following two conditions holds*

- (a) $G_{g=f}$ is bipartite, or
- (b) $g \equiv f$, $\sum_{v \in V} f(v) \equiv 0 \pmod{2}$ and G satisfies odd-cycle property,

then G has a (g, f) -factor if and only if G has a fractional (g, f) -factor.

Corollary 8.2.4. *Let G be a graph. If $g(v) < f(v)$ for all vertices $v \in V(G)$ or G is a bipartite graph. Then G has a (g, f) -factor if and only if G has a fractional (g, f) -factor.*

Now we present an algorithm for finding a fractional (g, f) -factor, which is deduced from the proof of Theorem 8.2.1. Let $L = v_1 v_2 \dots v_n$ be a *chain* of G such that $v_i v_{i+1} \in E(G)$ and the edges in L may be used more than once but *at most twice*. Similar to the definition of augmenting paths, we can define augmenting chains.

An *augmenting chain* of G with respect to h is an odd v -alternating chain $L_o(v_1, v)$ with $h(E_{v_1}) < g(v_1)$ and $h(E_v) < f(v)$, or an even v -alternating chain $L_e(v_1, v)$ with $h(E_{v_1}) < g(v_1)$ and $h(E_v) > g(v)$.

An Algorithm for Finding Fractional (g, f) -Factors

Step 1. Find a $(0, f)$ -factor $F = G_h$ of G (one may take F with $E(F) = \emptyset$) with indicator function h .

Step 2. Find an augmenting chain $L = v_1 v_2 \dots v_k$ with respect to h . If $L = L_o(v_1, v)$ where $v = v_k$ and $k = 2n$, then let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ (ε_i is defined as in the proof of Theorem 8.2.1); if $L = L_e(v_1, v)$ where $v = v_k$ and $k = 2n + 1$, then let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_4\}$. Set $h'(v_{2i-1} v_{2i}) = h(v_{2i-1} v_{2i}) + \varepsilon$, $h'(v_{2i} v_{2i+1}) = h(v_{2i} v_{2i+1}) - \varepsilon$, and $h'(e) = h(e)$ when $e \notin L$. If there are no augmenting chains with respect to h in G , then go to Step 4.

Step 3. Set $G_{h'} := G_{h'}$ and go to Step 2.

Step 4. If $U = \emptyset$, then $F = G_h$ is a fractional (g, f) -factor of G . Otherwise, $U \neq \emptyset$, and G has no fractional (g, f) -factors.

From the proof of Theorem 8.2.1, it is easy to see that if there are no augmenting chains with respect to h in G , then F is a fractional $(0, f)$ -factor with minimum η . If $U = \emptyset$, then $\eta(F) = 0$, and F is a fractional (g, f) -factor of G . If $U \neq \emptyset$, then $\eta(F) > 0$ and G has no fractional (g, f) -factors.

Since each search for an augmenting chain can be performed by breadth first search, this algorithm is of time complexity $O(g(V) \cdot |E|)$. Note that the augmenting chain can not be replaced by augmenting path in the algorithm.

The following theorem gives a degree condition for a graph to having a (g, f) -factor.

Theorem 8.2.3 (Liu and Zhang [368]). *Let G be a graph. If*

$$\frac{g(w)}{d_G(w)} \leq \frac{f(v)}{d_G(v)} \quad (8.6)$$

for all pairs of vertices $v, w \in V(G)$, then G has a fractional (g, f) -factor.

Proof. To prove the theorem, it suffices to verify condition (8.2) of Theorem 8.2.1. For any subset $S \subseteq V(G)$, let $T = \{v \mid v \in V(G) - S \text{ and } d_{G-S}(v) < g(v)\}$, $\bar{S} = V(G) - S$. Without loss of generality, we may assume that $0 \leq g(v) \leq f(v) \leq d_G(v)$. If $S = \emptyset$, then (8.2) holds trivially. So we assume $S \neq \emptyset$. Then (8.6) implies $f(S)d_G(T) - d_G(S)g(T) \geq 0$. Note that $d_G(\bar{S}) - d_{G-S}(\bar{S}) \leq d_G(S)$ and $d_G(S) \geq f(S)$. So

$$\begin{aligned} f(S) - (g(T) - d_{G-S}(T)) &\geq \frac{f(S)}{d_G(S)}(d_G(\bar{S}) - d_{G-S}(\bar{S})) - (g(T) - d_{G-S}(T)) \\ &= \frac{1}{d_G(S)}(f(S)d_G(T) - d_G(S)g(T)) \\ &\quad + \frac{1}{d_G(S)}(d_G(S) - f(S))d_{G-S}(T) \\ &\quad + \frac{f(S)}{d_G(S)}(d_G(V(G) - (S \cup T)) \\ &\quad - d_{G-S}(V(G) - (S \cup T))) \\ &\geq 0. \end{aligned}$$

Then $g(T) - d_{G-S}(T) \leq f(S)$ or (8.2) holds. Thus G has a fractional (g, f) -factor. $\square$

If $g(v) = f(v)$ for each $v \in V(G)$, then (8.6) holds if and only if $\frac{f(v)}{d_G(v)} = k$, where k is a real number. By Theorem 8.2.3 we have the following immediate assertion.

Corollary 8.2.5. *Let G be a graph and m an integer. If $mg(v) \leq d_G(v) \leq mf(v)$ for every vertex $v \in V(G)$, then G has a fractional (g, f) -factor.*

In particular, if $d_G(v) = mf(v)$, then G has a fractional f -factor. This yields an interesting fact that any r -regular graph has a fractional k -factor

for all $k \leq r$. However, Theorem 8.2.3 and Corollary 8.2.5 do not hold for (g, f) -factors. Actually, an r -regular graph with odd order has no k -factors for any k , $k < r$.

Suppose that G has a fractional (g, f) -factor. For a fixed edge $e \in E(G)$, when does G have a fractional (g, f) -factor containing or excluding e ? The following two theorems provide partial answers to this question. Without loss of generality, we assume that $0 \leq g(v) \leq f(v) \leq d_G(v)$ for all $v \in V(G)$.

Theorem 8.2.4 (Yang, Ma and Liu [607]). *Let G be a graph and $(f(v) - 1)d_G(w) \geq (d_G(v) - 1)g(w)$ holds for any $v, w \in V(G)$ and $v \neq w$. Then for any edge $e \in E(G)$, G has a fractional (g, f) -factor $F = G_h$ such that $h(e) = 1$.*

Proof. Let $e = xy$ be any edge of G and $G' = G - e$. Set $f'(x) = f(x) - 1$, $g'(x) = g(x) - 1$, $f'(y) = f(y) - 1$ and $g'(y) = g(y) - 1$. If $v \neq x, y$, then let $f'(v) = f(v)$, $g'(v) = g(v)$. Obviously, to prove the theorem, it is sufficient to verify that G' has a fractional (g', f') -factor. Then by Theorem 8.2.3, we need only to show that

$$\frac{g'(w)}{d_{G'}(w)} \leq \frac{f'(v)}{d_{G'}(v)} \quad (8.7)$$

holds for any $v, w \in V(G)$. Now we consider the following cases.

Case 1. $v = x$, $w = y$.

Since $(f(x) - 1)d_G(y) \geq (d_G(x) - 1)g(y)$, we have

$$\begin{aligned} f'(x)d_{G'}(y) &= (f(x) - 1)(d_G(y) - 1) \\ &\geq (d_G(x) - 1)g(y) - f(x) + 1 \\ &= d_{G'}(x)(g(y) - 1) + d_{G'}(x) - f(x) + 1 \\ &\geq d_{G'}(x)g'(y) + d_{G'}(x) - f(x) + 1 \\ &\geq d_{G'}(x)g'(y). \end{aligned}$$

Case 2. $v = x$, $w \neq y$.

In this case, we have

$$f'(x)d_{G'}(w) = (f(x) - 1)d_G(w) \geq (d_G(x) - 1)g(w) = d_{G'}(x)g'(w).$$

Case 3. $v \neq x$, $w = y$.

In this case, we have

$$\begin{aligned}
f'(v)d_{G'}(y) &= f(v)(d_G(y) - 1) \\
&= (f(v) - 1)d_G(y) + d_G(y) - f(v) \\
&\geq (d_G(v) - 1)g(y) + d_G(y) - f(v) \\
&= d_G(v)(g'(y) + 1) - g(y) + d_G(y) - f(v) \\
&= d_{G'}(v)g'(y) + d_G(v) - f(v) + d_G(y) - g(y) \\
&\geq d_{G'}(v)g'(y).
\end{aligned}$$

Case 4. $v \neq x, w \neq y$.

Then

$$\begin{aligned}
f'(v)d_{G'}(w) &= f(v)d_G(w) \\
&= (f(v) - 1)d_G(w) + d_G(w) \\
&\geq (d_G(v) - 1)g(w) + d_G(w) \\
&= d_G(v)g(w) + d_G(w) - g(w) \\
&= d_{G'}(v)g'(w) + d_G(w) - g(w) \\
&\geq d_{G'}(v)g'(w).
\end{aligned}$$

In all cases, we have verified (8.7). So, by Theorem 8.2.3, G' has a fractional (g', f') -factor. Let $h(e) = 1$. If $e' \in E(G)$ and $e' \neq e$, then let $h(e') = h'(e')$. Thus $F = G_h$ is the fractional (g, f) -factor required. $\square$

Theorem 8.2.5 (Yang, Ma and Liu [607]). *Let G be a graph. If, for any pair of vertices $v, w \in V(G)$ and $v \neq w$, $f(v)(d_G(w) - 1) \geq d_G(v)g(w)$ holds, then for any edge $e \in E(G)$, G has a fractional (g, f) -factor $F = G_h$ which excludes e , i.e., $h(e) = 0$.*

Proof. By Theorem 8.2.3, to prove the theorem, it suffices to verify that

$$f(v)d_{G'}(w) \geq g(w)d_{G'}(v) \quad (8.8)$$

holds for any $v, w \in V(G)$ with $v \neq w$. Now we consider the following four cases.

Case 1. $v = x, w = y$.

Then we have

$$\begin{aligned}
f(x)d_{G'}(y) &= f(x)(d_G(y) - 1) \geq d_G(x)g(y) \\
&= (d_{G'}(x) + 1)g(y) \geq d_{G'}(x)g(y).
\end{aligned}$$

Case 2. $v = x, w \neq y$.

Then

$$\begin{aligned}
f(x)d_{G'}(w) &= f(x)d_G(w) = f(x)(d_G(w) - 1) + f(x) \\
&\geq d_G(x)g(w) + f(x) \geq d_{G'}(x)g(w).
\end{aligned}$$

Case 3. $v \neq x$, $w = y$.

We have

$$f(v)d_{G'}(y) = f(v)(d_G(y) - 1) \geq d_G(v)g(y) = d_{G'}(v)g(y).$$

Case 4. $v \neq x$, $w \neq y$.

Then

$$\begin{aligned} f(v)d_{G'}(w) &= f(v)d_G(w) = f(v)(d_G(w) - 1) + f(v) \\ &\geq d_G(v)g(w) + f(v) \geq d_{G'}(v)g(w). \end{aligned}$$

In conclusion, (8.8) holds in all cases. The proof is completed. $\square$

Note that the conditions in Theorems 8.2.4 and 8.2.5 are only sufficient ones, but may not be necessary. The following result obtained by Zhang (2001) is analogous to Theorem 8.1.6.

Theorem 8.2.6 (Zhang [626]). *If G has a fractional (g, f) -factor, then it must have a fractional (g, f) -factor $F = G_h$ with the maximum sum of weights such that $h(e) \in \{0, \frac{1}{2}, 1\}$ for every edge $e \in E(G)$, where h is the indicator function of F .*

8.3 Parameters and fractional factors of graphs

In the previous two sections, the conditions for a graph G to have a fractional factor are mostly related to degrees. Now we present several sufficient conditions, such as binding number, toughness, isolated toughness, etc., for existence of fractional factors.

Recall that the binding number $bind(G)$ is defined as

$$bind(G) = \min \left\{ \frac{|N(X)|}{|X|} \mid \emptyset \neq X \subset V(G), N(X) \neq V(G) \right\}.$$

Anderson [54] gave a condition for which a graph G has a 1-factor. He showed that if $bind(G) \geq \frac{4}{3}$, then G has a 1-factor, and this bound is tight. Later Yu and Liu (2004) obtained a necessary and sufficient condition in terms of binding number, for a graph to have a fractional 1-factor.

Theorem 8.3.1 (Yu and Liu [608]). *Let G be a graph. Then G has a fractional 1-factor if and only if $bind(G) \geq 1$.*

Proof. By Theorem 8.1.10, we need to show that $i(G - S) \leq |S|$ for any $S \subseteq V(G)$. Otherwise, there exists a subset $S \subseteq V(G)$ such that $i(G - S) > |S|$. Obviously, $S \neq V(G)$. Since $bind(G) \geq 1$, we have $i(G) = 0$, $S \neq \emptyset$ and

$N(L(G - S)) \subseteq S$. Thus we have $|N(L(G - S))| \leq |S|$ and $N(L(G - S)) \neq V(G)$. Combining all the above, we have

$$\text{bind}(G) \leq \frac{|N(L(G - S))|}{|L(G - S)|} \leq \frac{|S|}{i(G - S)} < 1,$$

a contradiction.

Conversely, suppose G has a fractional 1-factor, we show that $\text{bind}(G) \geq 1$. Otherwise, there exists a proper subset X such that $\frac{|N(X)|}{|X|} < 1$, i.e., $|N(X)| < |X|$. Then $G[X]$ has at least one isolated vertex. Otherwise, we have $X \subseteq N(X)$ and then $|N(X)| \geq |X|$, a contradiction. Let $S = N(X) - X$. Then

$$\begin{aligned} i(G - S) &\geq |X - N(X)| = |X| - |N(X) \cap X| \\ &> |N(X)| - |N(X) \cap X| \\ &= |N(X) - X| = |S|. \end{aligned}$$

It follows from Theorem 8.1.10 that G has no fractional 1-factors, a contradiction. $\square$

In Chapter 3 we introduced the toughness of a graph G , $t(G)$, i.e., if G is not a complete graph, then

$$t(G) = \min \left\{ \frac{|S|}{\omega(G - S)} \mid S \subseteq V(G), \omega(G - S) \geq 2 \right\};$$

if G is complete, then we define $t(G) = \infty$. Many researchers have investigated the relationships between the toughness and factors. Below we present some results in this direction.

Theorem 8.3.2 (Liu and Zhang [371]). *Let G be a connected graph of order at least 2. Then G has a fractional perfect matching if $t(G) \geq 1$.*

Proof. If G is complete, obviously G has a fractional perfect matching as $|V(G)| \geq 2$. So we assume that G is not complete. Suppose that G has no fractional perfect matchings. By Theorem 8.1.10, there exists a subset S of $V(G)$ such that $i(G - S) > |S|$. Since G is connected, $S \neq \emptyset$. Thus $i(G - S) \geq 2$. Then

$$t(G) \leq \frac{|S|}{\omega(G - S)} \leq \frac{|S|}{i(G - S)} < 1,$$

a contradiction to $t(G) \geq 1$. $\square$

The result in Theorem 8.3.2 is sharp. To see this, consider the graph $G = K_m \vee (m + 1)K_1$, where m is an arbitrary positive integer. It is easy to verify

that $t(G) = \frac{m}{m+1} < 1$. Let $S = V(K_m)$. Then we have $i(G - S) > |S|$, and therefore, G has no fractional perfect matchings by Theorem 8.1.10. But we have $t(G) \rightarrow 1$ when $m \rightarrow +\infty$.

Liu and Zhang (2008) also obtained a sufficient condition for a graph to have a fractional k -factor.

Theorem 8.3.3 (Liu and Zhang [371]). *Let G be a graph and $k \geq 2$ be an integer. If $|V(G)| \geq k+1$ and $t(G) \geq k - \frac{1}{k}$, then G has a fractional k -factor.*

To study fractional factors, Yang, Ma and Liu [607] introduced a new parameter, *isolated toughness* of a graph G , denoted by $I(G)$, which is defined as

$$I(G) = \min \left\{ \frac{|S|}{i(G-S)} \mid S \subseteq V(G), i(G-S) \geq 2 \right\},$$

if G is not complete. Otherwise, $I(G) = \infty$. This new parameter turns out to be a nice tool to study fractional k -factors. This is not totally surprise for the case of fractional matchings if one notices Fractional Tutte's Theorem (i.e., Theorem 8.1.10). However, it is not obvious that $I(G)$ can be served as a nice condition for existence of k -factors as shown in the next theorem. The case of $k = 1$ was proved by Yang, Ma and Liu [607]; the case of $k = 2$ was settled by Ma and Liu [417], and the general case is due to Ma and Liu [418].

Theorem 8.3.4 (Ma and Liu [418]). *Let G be a graph with $\delta(G) \geq k$ and $I(G) \geq k$, where k is a positive integer. Then G has a fractional k -factor.*

Proof. If $k = 1$, from Theorem 8.1.10, it suffices to show that $i(G - S) \leq |S|$ for any $S \subseteq V(G)$. Without loss of generality, assume that G is connected. Otherwise, we consider each component instead. If $S = \emptyset$, then $i(G - S) = 0$ as $|V(G)| \geq 2$. So the inequality $i(G - S) \leq |S|$ holds. Now we assume that $S \neq \emptyset$. It is obvious that $i(G - S) \leq |S|$ if $i(G - S) = 1$. If $i(G - S) \geq 2$, then $i(G - S) \leq |S|$ follows from $I(G) \geq 1$. Thus G has a fractional 1-factor.

The case of $k = 2$ can be dealt similarly. So we assume that $k \geq 3$.

If G is a complete graph with $|V(G)| \geq k+1$, then let $h(e) = \frac{k}{|V(G)|-1}$ for each edge $e \in E(G)$. It is easy to see that G_h , the fractional factor of G with indicator function h , is a fractional k -factor of G . So we may assume that G is not complete. Suppose, to the contrary, that G has no fractional k -factors. Then by Corollary 8.2.2, there exists a subset $S_0 \subseteq V(G)$ such that

$$k|T_0| - d_{G-S_0}(T_0) > k|S_0|, \quad (8.9)$$

where $T_0 = \{v \mid v \in V(G) - S_0, d_{G-S_0}(v) < k\}$.

If $S_0 = \emptyset$, then we have $k|T_0| - d_{G-S_0}(T_0) = k|T_0| - k|T_0| = 0$ since $\delta(G) \geq k$, a contradiction to (8.9). Thus $S_0 \neq \emptyset$. If $G[T_0]$ is a complete graph, by (8.9), then $k|T_0| - d_{G-S_0}(T_0) = k > k|S_0|$ and thus $S_0 = \emptyset$, a contradiction. Let $U_0 = V(G) - (S_0 \cup T_0)$, $n = e_G(T_0, U_0)$ and $m = e_G(T_0) = e_G(T_0, T_0)$. Then (8.9) can be rewritten as

$$k|S_0| - k|T_0| + d_{G-S_0}(T_0) = k|S_0| - k|T_0| + 2m + n < 0. \quad (8.10)$$

Let A_1 be a maximal independent set of T_0 and $T_1 = T_0 - A_1$. Let A_i be a maximal independent set of T_{i-1} , $2 \leq i \leq k$. Since $\Delta(G[T_0]) < k$ and $G[T_0]$ is not a complete graph, $A_k = \emptyset$. Therefore, we need only to consider the case for $2 \leq i \leq k-1$. We require the following claims.

Claim 1. $|S_0| + |T_1| + e_G(U_0, A_1) \geq k|A_1|$.

Let $U_1 = \{u \mid u \in U_0, e_G(\{u\}, A_1) > 0\}$, $U_2 = \{u \mid u \in U_0, e_G(\{u\}, A_1) = 1\}$. Let $U_3 = U_1 - U_2$. Then $|U_1| \leq e_G(U_1, A_1) = e_G(U_0, A_1)$, $|U_3| \leq |U_1|$. Let $G_1 = G - (S_0 \cup T_1 \cup U_1)$. Then $i(G_1) \geq |A_1|$ as A_1 is a maximum independent set of T_0 .

Case 1. $i(G_1) \geq 2$.

Then we have $|S_0 \cup T_1 \cup U_1| \geq k \cdot i(G_1) \geq k|A_1|$ since $I(G) \geq k$. On the other hand, $|S_0 \cup T_1 \cup U_1| = |S_0| + |T_1| + |U_1| \leq |S_0| + |T_1| + e_G(U_0, A_1)$. Therefore, we have $|S_0| + |T_1| + e_G(U_0, A_1) \geq k|A_1|$, and thus Claim 1 holds.

Case 2. $i(G_1) = 1$.

In this case $|A_1| = 0$ or 1 as $i(G_1) \geq |A_1|$. If $|A_1| = 0$, then Claim 1 follows from $A_1 = T_0 = \emptyset = T_1$. If $|A_1| = 1$, let $A_1 = \{x\}$. Since $|S_0| + |T_1| + e_G(U_0, A_1) \geq d_G(x) \geq k$, we have $|S_0| + |T_1| + e_G(U_0, A_1) \geq k|A_1| = k$. So Claim 1 holds.

Case 3. $i(G_1) = 0$.

That is, $i(G - (S_0 \cup T_1 \cup U_1)) = 0$. Thus $A_1 = \emptyset$ and $T_0 = \emptyset$ since $i(G_1) \geq |A_1|$. So Claim 1 holds.

Claim 2. For each i , $2 \leq i \leq k-1$, we have

$$|S_0| + e_G(T_0 - T_{i-1}, A_i) + |T_i| + e_G(U_0, A_i) \geq k|A_i|.$$

If $A_i = \emptyset$, then $T_i = \emptyset$ and thus Claim 2 holds.

We assume that $A_i \neq \emptyset$ and let $X_i = N_G(A_i) \cap (T_0 - T_{i-1})$, $Y_i = N_G(A_i) \cap U_0$, $G_i = G - N_G(A_i)$. Clearly, A_i is a subset consisting of isolated vertices in G_i and $i(G_i) = i(G - N_G(A_i)) \geq |A_i|$. Now consider the following cases.

Case 1. $i(G_1) \geq 2$.

Since $I(G) \geq k$, we have $|S_0 \cup X_i \cup Y_i \cup T_i| \geq |N_G(A_i)| \geq k \cdot i(G_i) \geq k|A_i|$. On the other hand, $|S_0 \cup X_i \cup Y_i \cup T_i| \leq |S_0| + e_G(T_0 - T_{i-1}, A_i) + e_G(U_0, A_i) + |T_i|$. Therefore, $|S_0| + e_G(T_0 - T_{i-1}, A_i) + e_G(U_0, A_i) + |T_i| \geq k|A_i|$.

Case 2. $i(G_1) = 1$.

Since $A_i \neq \emptyset$, we have $|A_i| = 1$. Let $A_i = \{x_i\}$. Then $|S_0| + e_G(T_0 - T_{i-1}, A_i) + e_G(U_0, A_i) + |T_i| \geq d_G(x_i) \geq k = k|A_i|$. So Claim 2 holds.

By Claims 1 and 2, we have

$$\begin{aligned}
k|T_0| &= \sum_{i=1}^{k-1} k|A_i| \\
&\leq \sum_{i=1}^{k-1} |S_0| + \sum_{i=1}^{k-1} |T_i| + \sum_{i=2}^{k-1} e_G(T_0 - T_{i-1}, A_i) + \sum_{i=1}^{k-1} e_G(U_0, A_i) \\
&= (k-1)|S_0| + \sum_{i=1}^{k-1} |T_i| + m + n.
\end{aligned}$$

By (8.9) and (8.10), we have

$$k|S_0| + 2m + n < k|T_0| \leq (k-1)|S_0| + \sum_{i=1}^{k-1} |T_i| + m + n.$$

Thus

$$\sum_{i=1}^{k-1} |T_i| > |S_0| + m \geq m. \quad (8.11)$$

On the other hand,

$$m = e_G(T_0) = \sum_{1 \leq i < j \leq k-1} e_G(A_i, A_j) = \sum_{i=1}^{k-1} e_G(A_i, \bigcup_{j=i+1}^{k-1} A_j) = \sum_{j=1}^{k-1} e_G(A_j, T_j).$$

Since A_j is a maximal independent set of T_{j-1} , $e_G(A_j, \{x\}) \geq 1$ for every $x \in T_j$. Therefore, $e_G(A_j, T_j) \geq |T_j|$ and $m = \sum_{j=1}^{k-1} e_G(A_j, T_j) \geq \sum_{j=1}^{k-1} |T_j|$, a contradiction to (8.11). So the theorem is proved. $\square$

The result in Theorem 8.3.4 is best possible in the following sense. The isolated toughness condition $I(G) \geq k$ can not be replaced by $I(G) \geq k - \varepsilon$, where ε is any positive real number. We construct a graph H from K_{nk^2} , K_{k-1} and $(nk+1)K_1$ as follows. Let $V(K_{nk^2}) = \{x_1, x_2, \dots, x_{nk^2}\}$, $V((nk+1)K_1) = \{y_1, y_2, \dots, y_{nk+1}\}$ and $V(K_{k-1}) = \{z_1, z_2, \dots, z_{k-1}\}$, where n is any positive integer. Let the edge set of H be $E(H) = E(K_{nk^2}) \cup E(K_{k-1}) \cup \left(\bigcup_{i=1}^{nk+1} x_i y_i \right) \cup \{uv \mid u \in V(K_{k-1}), v \in V((nk+1)K_1)\}$. It is easy to see that $\delta(H) \geq k$ and $|V(H)| > k+1$. Let $S = V(K_{k-1})$. Then $\sum_{j=0}^{k-1} (k-j)p_j(H-S) = (nk+1)(k-1) > k(k-1) = k|S|$. So the graph H has no fractional k -factors by Corollary 8.2.3. It is not hard to see that isolated toughness of H is $I(H) = \frac{nk^2+k-1}{nk+1} = k - \frac{1}{nk+1}$. Therefore, for any given positive real number ε , there exists an integer n large enough (with respect

to ε) such that $I(H) = k - \frac{1}{nk+1} \geq k - \varepsilon$.

For the case that G has a fractional $[a, b]$ -factor, Ma and Liu (2006) obtained a sufficient condition in terms of isolated toughness.

Theorem 8.3.5 (Ma and Liu [418]). *Let G be a graph and $a < b$ be two positive integers. If $\delta(G) \geq I(G) \geq a - 1 + \frac{a}{b}$, then G has a fractional $[a, b]$ -factor.*

Next we discuss existence of fractional factors if some edges or vertices are removed.

Theorem 8.3.6 (Ma [416]). *Let G be a graph. If $\kappa(G) \geq 2k - 1$ and $I(G) \geq k \geq 1$, then the subgraph obtained by removing arbitrary $k - 1$ edges from G still has a fractional 1-factor.*

Proof. If $k = 1$, then the theorem follows from Theorem 8.3.4. So we assume that $k \geq 2$.

Suppose that G is a complete graph. If $|V(G)|$ is even, then $E(G)$ can be partitioned into $|V(G)| - 1$ 1-factors. If $|V(G)|$ is odd, then $E(G)$ can be partitioned into $\frac{1}{2}(|V(G)| - 1)$ Hamiltonian cycles. Since $|V(G)| \geq 2k$, the subgraph resulted from removing $k - 1$ edges from G has either a 1-factor or a Hamiltonian cycle. Note that a Hamiltonian cycle is also a fractional 1-factor (let $h(e) = \frac{1}{2}$ for every edge). So the theorem holds if G is complete.

Next suppose that G is not a complete graph. Let E_1 be the set of any $k - 1$ edges and $G' = G - E_1$. By Theorem 8.1.10, we need only to verify that $i(G' - S) \leq |S|$ holds for any $S \subseteq V(G')$. Note that $\kappa'(G) \geq \kappa(G) \geq 2k - 1$, so G' is connected, and thus $i(G') = 0$. If $S = \emptyset$, it is clear that $i(G' - S) \leq |S|$. If $S \neq \emptyset$, we consider the following three cases. Note that for any set $S \subseteq V(G)$, then $\omega(G - S) \geq i(G - S)$, and for any edge $e \in E(G)$, $\omega(G - e) \leq \omega(G) + 1$ and $i(G - e) \leq i(G) + 2$.

Case 1. $i(G - S) \leq 1$.

Suppose that S is not a cut set, i.e., $\omega(G - S) = \omega(G) = 1$. If $|S| \geq k$, then $i(G' - S) \leq k \leq |S|$ as $i(G' - S) = i((G - S) - E_1) \leq \omega(G - S) + k - 1 = k$. If $|S| = r \leq k - 1$, then $\kappa'(G) \geq \kappa(G) \geq 2k - 1$ and thus $\kappa'(G - S) \geq \kappa(G - S) \geq 2k - 1 - r$. Therefore

$$i(G' - S) \leq i(G - S) + 2k - 2 - (2k - 1 - r) = i(G - S) + r - 1 \leq r = |S|.$$

Suppose that S is a cut set, then $|S| \geq 2k - 1$ and $\omega(G - S) \geq 2$. So $i(G' - S) = i((G - S) - E_1) \leq i(G - S) + 2k - 2 \leq 2k - 1$. Thus we have $i(G' - S) \leq 2k - 1 \leq |S|$.

Case 2. $i(G - S) = 2$.

Then $\frac{|S|}{i(G - S)} \geq I(G) \geq k$, and so $|S| \geq 2k$. Thus,

$$\frac{|S|}{i(G' - S)} \geq \frac{|S|}{i(G - S) + 2k - 2} \geq \frac{2k}{2 + (2k - 2)} = 1.$$

Case 3. $i(G - S) \geq 3$.

Note that $k \geq 2$, so $|S| \geq i(G - S)k \geq i(G - S) + 2k - 1$. Then

$$\frac{|S|}{i(G' - S)} \geq \frac{|S|}{i(G - S) + 2k - 2} \geq \frac{i(G - S) + 2k - 1}{i(G - S) + 2k - 2} > 1,$$

and immediately we have $i(G' - S) < |S|$.

Therefore, $i(G' - S) \leq |S|$ holds in all cases, and it follows that G' has a fractional 1-factor. $\square$

To further investigate the relationships between isolated toughness and fractional factors, we introduce another new parameter $I'(G)$, which is a slight modification of $I(G)$, but it fits the study of fractional factors better. If G is not a complete graph, let

$$I'(G) = \min \left\{ \frac{|S|}{i(G - S) - 1} \mid S \subseteq V(G), i(G - S) \geq 2 \right\};$$

otherwise, let $I'(G) = \infty$.

We establish the link between $I(G)$ and $I'(G)$ in the next result.

Lemma 8.3.1 (Ma and Liu [417]). *Let G be a graph and k an integer. If $|V(G)| \geq k + 2$, then $I(G) \geq k$ if and only if $I'(G - S) \geq k$ for any $S \subseteq V(G)$ with $|S| = k$.*

Proof. Suppose that $I(G) \geq k$. Choose a subset S of $V(G)$ such that $|S| = k$, and let $R \subseteq V(G - S)$ be an arbitrary set. We have

$$|S \cup R| \geq k \cdot i(G - (S \cup R)).$$

Then

$$|R| \geq k(i((G - S) - R) - 1)$$

since $|S \cup R| = |S| + |R|$. Therefore, $I'(G - S) \geq k$.

Conversely, assume $I'(G - S) \geq k$ for any $S \subseteq V(G)$ with $|S| = k$. Choose any set $R \subseteq V(G)$ such that $i(G - R) \geq 2$.

We claim $|R| \geq k$, otherwise, if $|R| < k$, then there exists a subset $S_0 \subseteq V(G)$, such that $|S_0| = k$, $R \subset S_0$ and $i(G - S_0) \geq 2$. So

$$k \leq I'(G - S_0) \leq \frac{|\emptyset|}{i((G - S_0) - \emptyset) - 1} < k,$$

a contradiction. Now we choose a set $S \subseteq R$ such that $|S| = k$. Then

$$k \leq I'(G - S) \leq \frac{|R - S|}{i((G - S) - (R - S)) - 1} = \frac{|R - S|}{i(G - R) - 1}.$$

Thus

$$|R - S| \geq k(i(G - R) - 1),$$

and then,

$$|R| \geq k \cdot i(G - R).$$

So we have $\frac{|R|}{i(G-R)} \geq k$.

From the above discussion, for any set $R \subseteq V(G)$ with $i(G - R) \geq 2$, then we have $|R| \geq k$ and $\frac{|R|}{i(G-R)} \geq k$. Hence $I(G) \geq k$. $\square$

Lemma 8.3.1 plays an important role in our study of fractional k -factors. In many instances, we can improve the results by replacing the parameter $I(G)$ by $I'(G)$. The following corollary follows from Lemma 8.3.1 and Theorem 8.3.4.

Corollary 8.3.1. *Let G be a graph of order at least 3. If $I'(G - \{v\}) \geq 1$ for any vertex $v \in V(G)$, then G has a fractional 1-factor.*

Theorem 8.3.6 is a sufficient condition for a graph to have fractional factors after removing some edges. Next, we consider the action of removing vertices. We need a lemma first.

Lemma 8.3.2 (Ma and Liu [417]). *Let G be a connected non-complete graph. If $I'(G) > 1$ and $|V(G)| \geq 3$, then G has a fractional 1-factor.*

Proof. Suppose that G has no fractional 1-factors. Then by Theorem 8.1.10, there exists a nonempty subset $S_0 \subseteq V(G)$ such that $i(G - S_0) > |S_0|$. However, $i(G - S_0) - |S_0| \equiv 0 \pmod{2}$ or $i(G - S_0) - |S_0| - 1 \equiv 0 \pmod{2}$. Therefore, $i(G - S_0) = |S_0| + 2k$ or $i(G - S_0) = |S_0| + 2k - 1$ ($k = 1, 2, \dots$). Thus

$$I'(G) \leq \frac{|S_0|}{i(G - S_0) - 1} = \frac{|S_0|}{|S_0| + 2k - 1} < 1,$$

or

$$I'(G) \leq \frac{|S_0|}{i(G - S_0) - 1} = \frac{|S_0|}{|S_0| + 2k - 2} \leq 1.$$

Either of the above two inequalities yields a contradiction to $I'(G) > 1$. So G has a fractional 1-factor. $\square$

The condition $I'(G) > 1$ in Lemma 8.3.2 can not be weakened to $I'(G) \geq 1$. As an example, consider a path of order 3, P_3 . So $I'(P_3) = 1$, but P_3 has no fractional 1-factors.

Theorem 8.3.7 (Ma and Liu [417]). *Let G be a graph. If $\kappa(G) \geq k$ and $I(G) \geq k \geq 1$, then the subgraph obtained by removing arbitrary $k - 1$ vertices from G has a fractional 1-factor.*

Proof. If $k = 1$, then the theorem follows from Theorems 8.1.10 and 8.3.4.

Assume $k \geq 2$. Since $\kappa(G) \geq k$, we have $|V(G)| \geq k + 1$. If G is a complete graph, then obviously the subgraph of G by removing arbitrary $k - 1$ vertices is also a complete graph. So it has a fractional 1-factor.

Suppose that G is not complete. Let U be the set of $k - 1$ vertices and $G' = G - U$. By Theorem 8.1.10, we need only to show that $i(G' - S) \leq |S|$ holds for any $S \subseteq V(G')$. Since $\kappa(G) \geq k$, G' is connected. If $S = \emptyset$, we have $i(G' - S) = 0$ and thus $i(G' - S) \leq |S|$. If $S \neq \emptyset$, we consider the following two cases.

Case 1. $i(G' - S) \leq 1$.

Since $S \neq \emptyset$, we have $|S| \geq 1$. Note that $i(G' - S) \leq 1$, so $i(G' - S) \leq |S|$.

Case 2. $i(G' - S) \geq 2$.

Since $I(G) \geq k > k - 1$, we have $I'(G - S) \geq k - 1$ for any $S \subseteq V(G)$ with $|S| = k - 1$ by Lemma 8.3.1. So $I'(G - S) > 1$ if $k \geq 3$, and the theorem follows from Lemma 8.3.2. Next we prove the theorem for $k = 2$. It suffices to show that the graph $G' = G - \{v\}$ has a fractional 1-factor for any $v \in V(G)$. Since $i(G' - S) \geq 2$, i.e., $i(G - \{v\} - S) \geq 2$, we have

$$2 \leq I(G) \leq \frac{|S \cup \{v\}|}{i(G - (S \cup \{v\}))} = \frac{|S| + 1}{i(G' - S)}.$$

So $|S| \geq 2i(G' - S) - 1$. Note that $i(G' - S) \geq 2$, thus $|S| \geq i(G' - S)$.

In all cases we show $i(G' - S) \leq |S|$ for any $S \subseteq V(G')$, so by Theorem 8.1.10, G' has a fractional 1-factor. $\square$

To see that the theorem is best possible, note that if G is a complete graph with order $k + 1$, then the subgraph of G by removing k vertices has no fractional 1-factors. Additionally, suppose H is a complete graph with order $2k$ and let $V(H) = \{x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_k\}$. We add two new vertices u and v to H with extra edges $\{ux_i \mid 1 \leq i \leq k\}$ and $\{vy_i \mid 1 \leq i \leq k\}$. Denote this new graph by G . It is not hard to see that $\kappa(G) = k$ and $I(G) = k$. Now we delete k vertices $x_1, x_2, \dots, x_k$ (or $y_1, y_2, \dots, y_k$) from G , then the remaining subgraph has no fractional 1-factors. This implies that the number $k - 1$ in Theorem 8.3.7 can not move up to k .

Cai and Liu (2006) gave a sufficient condition using independent number for existence of fractional factors.

Theorem 8.3.8 (Cai and Liu [101]). *Let G be a graph and f be a non-negative integer-valued function defined on $V(G)$ such that $0 \leq a \leq f(v) \leq b$ for every $v \in V(G)$. If $\delta(G) \geq b$ and $\alpha(G) \leq \frac{4a(\delta-b)}{(b+1)^2}$, then G has a fractional f -factor.*

8.4 Maximum and minimum fractional (g, f) -factors

If a graph G has more than one fractional (g, f) -factors, how shall we determine which one is the maximum or minimum? If it is, then what properties does it possess? In this section, we attempt to answer these questions.

Suppose that $F = G_h$ is a fractional (g, f) -factor of G , where h is the indicator function. We define (g, f) -defect of F as follows

$$\text{def}(F) = \sum_{v \in V(G)} (f(v) - h(E_v)) = \sum_{v \in U} (f(v) - h(E_v)),$$

where $U = \{v \mid v \in V(G), h(E_v) < f(v)\}$. Define (g, f) -defect of G to be

$$\text{def}(G) = \min\{\text{def}(F) \mid F \text{ is a fractional } (g, f)\text{-factor of } G\}.$$

If F is a fractional (g, f) -factor of G such that $\text{def}(F) = \text{def}(G)$, then F is called a *maximum* fractional (g, f) -factor of G .

Analogue to the definition of augmenting path and augmenting chain, we introduce two new similar terms, expanding path and expanding chain. They are slight modifications from the former ones, but they are suitable for the discussion in this section.

Let $F = G_h$ be a fractional (g, f) -factor of G with indicator function h . A *v-alternating path* with respect to h is a path $P(v_1, v) = v_1 v_2 \dots v_k (= v)$ such that $h(v_{2i-1} v_{2i}) < 1$ and $h(v_{2i} v_{2i+1}) > 0$ for each i . The path $P(v_1, v)$ is *even* or *odd* according to k is odd or even, respectively. We denote an odd (even) v -alternating path from v_1 to v by $P_o(v_1, v)$ ($P_e(v_1, v)$). An *expanding path* of G with respect to h is an odd v -alternating path $P_o(v_1, v)$ with $h(E_{v_1}) < f(v_1)$ and $h(E_v) < f(v)$. Similarly, an *expanding chain* of G with respect to h is an odd v -alternating chain $L_o(v_1, v) = v_1 v_2 \dots v_{2k} (= v)$ with $h(v_{2i-1} v_{2i}) < 1$, $h(v_{2i} v_{2i+1}) > 0$, $h(E_{v_1}) < f(v_1)$ and $h(E_v) < f(v)$. Recall that in a chain the edges may be used more than once but at most twice. Next we give a necessary and sufficient condition for a graph G to have a maximum fractional (g, f) -factor.

Theorem 8.4.1 (Liu and Zhang [372]). *Let G be a graph and $F = G_h$ be a fractional (g, f) -factor. Then F is a maximum fractional (g, f) -factor if and only if there are no expanding chains with respect to h in G .*

Proof. Suppose that $F = G_h$ is a maximum fractional (g, f) -factor of G . Then we have $\text{def}(G) = \text{def}(F) = \sum_{v \in U} (f(v) - h(E_v))$. That is, F minimizes the quantity

$$\text{def}(F) = \sum_{v \in U} (f(v) - h(E_v)),$$

where $U = \{v \mid v \in V(G), h(E_v) < f(v)\}$. We show that there are no expanding chains in G . Otherwise, if there exists an expanding chain $L_o(v_1, v) =$

$v_1 v_2 \dots v_{2k}(=v)$ with respect to h such that $\{v_1, v\} \subseteq U$, then let

$$\begin{aligned}\varepsilon_1 &= \min_i \{1 - h(v_{2i-1} v_{2i})\}, \\ \varepsilon_2 &= \min_i \{h(v_{2i} v_{2i+1})\}.\end{aligned}$$

If $v_1 \neq v$, then let $\varepsilon_3 = \min\{f(v_1) - h(E_{v_1}), f(v) - h(E_v)\}$; if $v_1 = v$, let $\varepsilon_3 = \frac{1}{2} \min\{f(v_1) - h(E_{v_1}), f(v) - h(E_v)\}$.

Set

$$\varepsilon = \frac{1}{2} \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}.$$

Next we construct a new indicator function h' . Let $h'(v_{2i-1} v_{2i}) = h(v_{2i-1} v_{2i}) + \varepsilon$, $h'(v_{2i} v_{2i+1}) = h(v_{2i} v_{2i+1}) - \varepsilon$, and $h'(e) = h(e)$ if $e \notin L_o(v_1, v)$. It is easy to verify that $\text{def}(G_{h'}) < \text{def}(F)$, a contradiction (see Fig. 8.6).

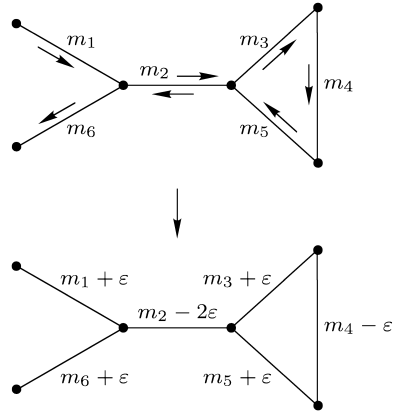


Fig. 8.6 Illustration of the change of an expanding chain.

Conversely, if there are no expanding chains with respect to h in G , we show that $F = G_h$ is a maximum fractional (g, f) -factor of G . By the definition, it suffices to show that $\text{def}(G) = \text{def}(F)$. If F is a fractional f -factor, then $\text{def}(G) = \text{def}(F) = 0$. Otherwise, let $U = \{v \mid v \in V(G), h(E_v) < f(v)\}$. We have $U \neq \emptyset$ and $\text{def}(F) > 0$. Now we define two sets:

$S^* = \{v \mid \text{there is an odd } v\text{-alternating path } P_o(v_1, v) \text{ in } G \text{ and } v_1 \in U\},$

$D = \{v \mid v \notin U, \text{ there exists an even } v\text{-alternating path } P_e(v_1, v) \text{ in } G \text{ and } v_1 \in U\}.$

Since there are no expanding chains with respect to h in G , $v \notin U$ or $h(E_v) = f(v)$ for all $v \in S^* \cup D$. It is easy to see that if $h(vw) > 0$, then $v \in S^*$ implies $w \in D \cup U$. If $vw \in E(G)$ and $h(vw) < 1$, then $v \in D \cup U$ implies $w \in S^*$. Now we show that $S^* \cap D = \emptyset$. Otherwise, if $v \in S^* \cap D$, then there exists an odd alternating path $P_o(v_1, v) = v_1 v_2 \dots v_{2k}(=v)$ and

an even alternating path $P_e(w_1, v) = w_1 w_2 \dots w_{2k+1} (= v)$, where $v_1, w_1 \in U$. Thus $P_o(v_1, v) \cup P_e(w_1, v)$ is an expanding chain, a contradiction.

Let

$$\eta(S) = f(T) - d_{G-S}(T) - f(S)$$

for any $S \subseteq V(G)$, where $T = \{v \mid v \in V(G) - S \text{ and } d_{G-S}(v) \leq f(v)\}$. Then for any subset $S \subseteq V(G)$ and any fractional (g, f) -factor H , we have

$$\eta(S) \leq \text{def}(H). \quad (8.12)$$

In particular,

$$\eta(S^*) \leq \text{def}(F). \quad (8.13)$$

Next, we show that $\eta(S^*) \geq \text{def}(F)$. Let $E_1 = \{e \mid e \in E(F), h(e) = 1\}$ and $F_1 = G[E_1]$. Then we have

$$d_{G-S^*}(v) = d_{F_1}(v) - e'_{F_1}(\{v\}, S^*) \leq h(E_v) - e'_{F_1}(\{v\}, S^*) \leq f(v)$$

for any $v \in D \cup U$. Let $T^* = \{v \mid v \in V(G) - S^* \text{ and } d_{G-S^*}(v) \leq f(v)\}$, then $D \cup U \subseteq T^*$. Thus

$$\begin{aligned} f(T^*) - d_{G-S^*}(T^*) &\geq f(D \cup U) - d_{G-S^*}(D \cup U) \\ &= f(D \cup U) - d_{F_1}(D \cup U) + e'_{F_1}(D \cup U, S^*) \\ &\geq f(U) - d_{F_1}(U) + f(D) - d_{F_1}(D) + e'_{F_1}(D \cup U, S^*). \end{aligned} \quad (8.14)$$

Note that if $v \in S^*$ and $h(vw) > 0$, then $w \in D \cup U$. Therefore,

$$\begin{aligned} \sum_{v \in S^*} h(E_v) &= h(E_G(S^*, V(G) - S^*)) = f(S^*) = h(E_F(S^*, V(G) - S^*)) \\ &= h(E_F(S^*, D \cup U)) = e'_F(S^*, D \cup U). \end{aligned} \quad (8.15)$$

Let $E_2 = \{e \mid e \in E(F), h(e) < 1\}$ and $F_2 = G[E_2]$. Then by (8.14) and (8.15), we have

$$\begin{aligned} f(T^*) - d_{G-S^*}(T^*) &\geq f(D \cup U) - d_{F_1}(D \cup U) + e'_{F_1}(D \cup U, S^*) \\ &= f(D \cup U) - d_{F_1}(D \cup U) + e'_{F_2}(D \cup U, S^*) \\ &\quad - e'_{F_2}(D \cup U, S^*) + e'_{F_1}(D \cup U, S^*) \\ &= f(D \cup U) - d_{F_1}(D \cup U) + e'_F(D \cup U, S^*) \\ &\quad - e'_{F_2}(D \cup U, S^*) \\ &= f(D) - d_{F_1}(D) - e'_{F_2}(D, S^*) + f(U) \\ &\quad - d_{F_1}(U) - e'_{F_2}(U, S^*) + f(S^*) \\ &= f(D) - \sum_{v \in D} h(E_v) + f(U) - \sum_{v \in U} h(E_v) + f(S^*) \\ &\geq \text{def}(F) + f(S^*). \end{aligned} \quad (8.16)$$

Note that in the above arguments, we also use $f(D) - \sum_{v \in D} h(E_v) = 0$, $f(U) - \sum_{v \in U} h(E_v) = \text{def}(F)$ and $e'_F(D \cup U, S^*) = f(S^*)$. Therefore

$$\eta(S^*) = f(T) - d_{G-S^*}(T) - f(S^*) \geq \text{def}(F). \quad (8.17)$$

It follows from (8.13), (8.17) and (8.12) that

$$\eta(S^*) = \text{def}(F) = \text{def}(G).$$

The proof is complete. $\square$

Since each search of an expanding chain can be performed by the breadth first search in time $O(|E|)$ and the corresponding expanding lowers the value $\max\{0, f(v) - h(E_v)\}$ for at least one vertex $v \in V(G)$, we have a very simple algorithm for finding a maximum fractional (g, f) -factor with time complexity $O(f(V) \cdot |E|)$.

If $P = v_0 v_1 \dots v_k$ is a path with edge set $\{e_1, e_2, \dots, e_k\}$ where $e_i = v_{i-1} v_i$, $1 \leq i \leq k$, then we also write $P = e_1 e_2 \dots e_k$ for convenience. Let $F = G_h$ be a fractional (g, f) -factor of G with indicator function h . Let $E_h^* = \{e \mid 0 < h(e) < 1\}$, and $\mathcal{F}_{\max} = \{F \mid F \text{ is a maximum fractional } (g, f)\text{-factor of } G\}$. If $|E_h^*| = \min\{|E_{h'}^*| \mid G_{h'} \in \mathcal{F}_{\max}\}$, then we say that $|E_h^*|$ is minimum in $\mathcal{F}_{\max}$, or simply $|E_h^*|$ is minimum, and denote $F^* = G[E_h^*]$.

Theorem 8.4.2 (Liu and Zhang [372]). *Let G be a graph and $F = G_h \in \mathcal{F}_{\max}$ with $|E_h^*|$ minimum. Then the subgraph $F^* = G[E_h^*]$ is a disjoint union of odd cycles.*

Proof. Let $F = G_h \in \mathcal{F}_{\max}$ with $|E_h^*|$ minimum. Then we have the following claims.

Claim 1. F^* has no even cycles.

Otherwise, assume that F^* has an even cycle $C = e_1 e_2 \dots e_{2l}$. Let

$$\varepsilon_1 = \min_{1 \leq i \leq 2l} \{h(e_i)\},$$

$$\varepsilon_2 = \min_{1 \leq i \leq 2l} \{1 - h(e_i)\},$$

and

$$\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}.$$

If $\varepsilon = \varepsilon_1$, without loss of generality, we assume $h(e_1) = \varepsilon$. Let p be a function defined on $E(G)$ which takes alternatively -1 and 1 on edges in $E(C)$ with $p(e_1) = -1$ and takes 0 on edges in $E(G) - E(C)$. Let $h' = h + \varepsilon p$, then $F' = G_{h'} \in \mathcal{F}_{\max}$ has $|E_{h'}^*| < |E_h^*|$, a contradiction.

If $\varepsilon = \varepsilon_2$, without loss of generality, say $1 - h(e_1) = \varepsilon$. Let p be a function defined on $E(G)$ which takes alternatively 1 and -1 on edges in $E(C)$ with

$p(e_1) = 1$ and takes 0 on edges in $E(G) - E(C)$. Let $h' = h + \varepsilon p$, then $F' = G_{h'} \in \mathcal{F}_{\max}$ has $|E_{h'}^*| < |E_h^*|$, also a contradiction. Thus Claim 1 is proved.

Claim 2. $d_{F^*}(v) \geq 2$ for any $v \in V(F^*)$.

Otherwise, suppose that there is a vertex $v_0 \in V(F^*)$ such that $d_{F^*}(v_0) = 1$. Then $h(E_{v_0}) < f(v_0)$. Let $P(v_0, v_k) = e_1 e_2 \dots e_k$ be a longest path in F^* starting from v_0 . If k is even, we have $d_{F^*}(v_k) = 1$. Let $\varepsilon = \min\{h(e_1), h(e_2), \dots, h(e_k)\}$. Without loss of generality, we may assume that $h(e_{i_0}) = \varepsilon$. Let $h'(e_i) = h(e_i) - \varepsilon$ if $i \equiv i_0 \pmod{2}$, $h'(e_i) = h(e_i) + \varepsilon$ if $i \equiv i_0 + 1 \pmod{2}$, and $h'(e) = h(e)$ for the remaining edges in $E(G)$. Then $F' = G_{h'} \in \mathcal{F}_{\max}$ has $|E_{h'}^*| < |E_h^*|$, contradicting to the definition of h . So k must be odd. Since F is a maximum fractional (g, f) -factor, there exist no expanding chains with respect to h by Theorem 8.4.1. Hence, $h(E_{v_k}) = f(v_k)$. This implies that there are two vertices on the path P adjacent to v_k , and therefore, an odd cycle is formed. Then the chain, which traverses the edges on the cycle once and the other edges of the path twice, is an expanding chain with respect to h , contradicting to $F \in \mathcal{F}_{\max}$ and Theorem 8.4.1. Thus Claim 2 is proved.

Claim 3. If F^* has an odd cycle C , then C must be an entire component of F^* .

Otherwise, let $C = e_1 e_2 \dots e_{2l+1}$ be an odd cycle of F^* with a vertex $v \in V(C)$ of degree greater than two in F^* . Suppose that e'_1 is an edge not in $E(C)$ and incident with v in F^* . Starting at e'_1 , we choose a longest path in $F^* - E(C)$. By Claim 1, the path can not return to a vertex of $V(C) - \{v\}$. By Claim 2, we eventually must obtain another cycle C' (also odd) which is connected with the cycle C by a path P (possibly of length 0). Let

$$\begin{aligned}\varepsilon_1 &= \min\{h(e) \mid e \in E(C \cup C')\}, \\ \varepsilon_2 &= \min\{1 - h(e) \mid e \in E(C \cup C')\}, \\ \varepsilon_3 &= \min\{h(e) \mid e \in E(P)\}, \\ \varepsilon_4 &= \min\{1 - h(e) \mid e \in E(P)\},\end{aligned}$$

and

$$\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \frac{1}{2}\varepsilon_3, \frac{1}{2}\varepsilon_4\}.$$

If $\varepsilon = \varepsilon_1$, we may assume that $h(e_1) = \varepsilon$ without loss of generality. Let p be a function defined on $E(G)$ which takes alternatively -1 and 1 on edges in $E(C \cup C')$ with $p(e_1) = -1$, takes -2 and 2 on edges in $E(P)$ and takes 0 on edges in $E(G) - E(C \cup C' \cup P)$ such that $p(E(G)) = 0$. One can easily check that such a function is feasible since both C and C' are odd cycles. Let $h' = h + \varepsilon p$, then $F' = G_{h'} \in \mathcal{F}_{\max}$ has $|E_{h'}^*| < |E_h^*|$, contradicting to the choice of h .

Similarly, it results a contradiction if $\varepsilon = \varepsilon_2, \frac{1}{2}\varepsilon_3$ or $\frac{1}{2}\varepsilon_4$. Thus Claim 3 is proved.

Combining Claims 1, 2 and 3, we conclude that every component of F^* is an odd cycle. $\square$

As a consequence of Theorem 8.4.2, we see that maximum fractional (g, f) -factors have a nice property, i.e., there exists a maximum fractional (g, f) -factor with indicator function only using values $0, \frac{1}{2}$ or 1 .

There are similar statements for fractional $[a, b]$ -factors, fractional k -factors, etc., we do not include it here. The following corollary follows from Theorem 8.4.2.

Corollary 8.4.1. *Let G be a graph and $F = G_h \in \mathcal{F}_{\max}$. Then $h(E(G))$ is an integer.*

Theorem 8.4.3 (Liu and Zhang [372]). *Suppose that G has a fractional (g, f) -factor. Then the maximum fractional (g, f) -factors with $|E_h^*|$ minimum are maximum fractional (g, f) -factors with minimum number of edges.*

Proof. Let $F = G_h$ be a maximum fractional (g, f) -factor with $|E_h^*|$ minimum. Let $m = h(E(G))$. Then by Corollary 8.4.1, m is an integer. Let

$$m_1 = |\{e \in E(G) \mid h(e) = 1\}|$$

and

$$m_2 = \left| \left\{ e \in E(G) \mid h(e) = \frac{1}{2} \right\} \right|.$$

We have $|F| = m_1 + m_2$, $m = m_1 + \frac{1}{2}m_2$. So

$$|F| = m + \frac{1}{2}m_2.$$

As $m_2 = |E_h^*|$ is minimized, $|F|$ is the minimum. $\square$

Now we begin to discuss the minimum fractional (g, f) -factor of a graph G . Suppose that $F = G_h$ is a fractional (g, f) -factor of G with indicator function h . We call $h(E(G))$ the *value* of F . The (g, f) -*surplus* of F , denoted by $sur(F)$, is defined as

$$sur(F) = \sum_{v \in V(G)} (h(E_v) - g(v)) = \sum_{v \in U} (h(E_v) - g(v)),$$

where $U = \{v \mid v \in V(G), h(E_v) > g(v)\}$. The (g, f) -*surplus* of G is defined to be

$$sur(G) = \min\{sur(F) \mid F \text{ is a fractional } (g, f)\text{-factor of } G\}.$$

F is a *minimum* fractional (g, f) -factor of G if F is a fractional (g, f) -factor such that $\text{sur}(F) = \text{sur}(G)$.

A *v-staggered* path with respect to h is a path $P(v_1, v) = v_1 v_2 \dots v_k (= v)$ of G , such that $h(v_{2i-1} v_{2i}) > 0$ and $h(v_{2i} v_{2i+1}) < 1$ ($i = 1, 2, \dots$). The path $P(v_1, v)$ is *even* or *odd* according to k is odd or even, respectively. We denote an odd (even) *v-staggered* path from v_1 to v by $P_o(v_1, v)$ ($P_e(v_1, v)$). A *reducing path* of G with respect to h is an odd *v-staggered* path $P_o(v_1, v)$ with $h(E_{v_1}) > g(v_1)$ and $h(E_v) > g(v)$. Similarly, a *reducing chain* of G with respect to h is an odd *v-staggered* chain $L_o(v_1, v) = v_1 v_2 \dots v_{2k} (= v)$ with $h(v_{2i-1} v_{2i}) > 0$, $h(v_{2i} v_{2i+1}) < 1$, $h(E_{v_1}) > g(v_1)$ and $h(E_v) > g(v)$, where a chain $v_1 v_2 \dots v_n$ means $v_i v_{i+1} \in E(G)$ and the edge may be used more than once but at most twice. Note that $v_1 = v$ is allowed.

Next we present necessary and sufficient conditions for a graph to have minimum fractional (g, f) -factors. The proof of the first theorem is similar to that of Theorem 8.4.1, so we omit the proof.

Theorem 8.4.4 (Liu and Li [356]). *Let G be a graph and $F = G_h$ be a fractional (g, f) -factor with indicator function h . Then F is a minimum fractional (g, f) -factor if and only if there are no reducing chains with respect to h in G .*

The next result reveals the relationship between a minimum fractional (g, f) -factor and its value.

Theorem 8.4.5 (Liu and Li [356]). *For a graph G , a subgraph F is a minimum fractional (g, f) -factor of G if and only if F is a fractional (g, f) -factor of G with minimum value.*

Proof. Suppose that F is a fractional (g, f) -factor of G with minimum value. Then we have $\text{sur}(F) = \text{sur}(G)$. Otherwise, there exists a minimum fractional (g, f) -factor $F' = G_{h'}$ such that $\text{sur}(G) = \text{sur}(F') < \text{sur}(F)$. Then

$$\text{sur}(F') = \text{sur}(G) = \sum_{v \in V(G)} (h'(E_v) - g(v)) < \sum_{v \in V(G)} (h(E_v) - g(v)).$$

Thus

$$\sum_{v \in V(G)} h'(E_v) < \sum_{v \in V(G)} h(E_v),$$

and therefore,

$$h'(E(G)) = \frac{1}{2} \sum_{v \in V(G)} h'(E_v) < \frac{1}{2} \sum_{v \in V(G)} h(E_v) = h(E(G)),$$

a contradiction.

Conversely, suppose that F is a minimum fractional (g, f) -factor of G . We show that F has the minimum value. Otherwise, there exists a fractional

(g, f) -factor $F' = G_{h'} (\neq F)$ of G with minimum value. From the arguments in the sufficient part, then F' is a minimum fractional (g, f) -factor. Thus

$$\text{sur}(F') = \text{sur}(G) = \sum_{v \in V(G)} (h'(E_v) - g(v)) = \sum_{v \in V(G)} (h(E_v) - g(v)).$$

Then

$$\sum_{v \in V(G)} h'(E_v) = \sum_{v \in V(G)} h(E_v),$$

so we have

$$h'(E(G)) = \frac{1}{2} \sum_{v \in V(G)} h'(E_v) = \frac{1}{2} \sum_{v \in V(G)} h(E_v) = h(E(G)),$$

a contradiction. The proof is completed. $\square$

Suppose that $F = G_h$ is a fractional (g, f) -factor of G . Let $E_h^* = \{e \mid 0 < h(e) < 1\}$ and

$$\mathcal{F}_{\min} = \{F \mid F \text{ is a minimum fractional } (g, f)\text{-factor of } G\}.$$

If $|E_h^*| = \min\{|E_{h'}^*| \mid G_{h'} \in \mathcal{F}_{\min}\}$, then we say that $|E_h^*|$ is *minimum* in $\mathcal{F}_{\min}$, or simply $|E_h^*|$ is *minimum*. Next, we study the properties of fractional (g, f) -factors in $\mathcal{F}_{\min}$ with minimum number of edges. We need a lemma first.

Lemma 8.4.1 ([356]). *Let G be a graph and $F = G_h \in \mathcal{F}_{\min}$ with $|E_h^*|$ minimum. Then each component of $F^* = G[E_h^*]$ is an odd cycle and $h(e) = \frac{1}{2}$ for every $e \in E_h^*$.*

Suppose that $F = G_h$ is a minimum fractional (g, f) -factor of G . Let C_1 and C_2 be two disjoint odd cycles of G and P be a path with $|E(P)| \geq 1$ in G joining C_1 and C_2 , i.e., P has one end vertex in C_1 and the other in C_2 , and each internal vertex of P is not in $C_1 \cup C_2$. If $h(e) = \frac{1}{2}$ for every $e \in C_1 \cup C_2$ and h takes alternatively 0 and 1 on edges of P , then the chain which traverses the edges of $C_1 \cup C_2$ once and the edges of P twice is called an *even decreasing chain* with respect to $F^* = G[E_h^*]$.

Theorem 8.4.6 (Liu and Li [356]). *Let G be a graph and $F = G_h \in \mathcal{F}_{\min}$. Then $|E_h^*|$ is minimum if and only if both (a) and (b) hold.*

- (a) *Every component of $F^* = G[E_h^*]$ is an odd cycle and $h(e) = \frac{1}{2}$ for every $e \in E_h^*$;*
- (b) *there are no even decreasing chains with respect to $F^* = G[E_h^*]$ in G .*

Proof. Suppose that $F = G_h$ is a minimum fractional (g, f) -factor of G with $|E_h^*|$ minimum. Then (a) follows from Lemma 8.4.1. Now we prove (b). Suppose that there is an even decreasing chain with respect to F^* . Denote the two

odd cycles by $C_1 = x_1x_2 \dots x_{2k+1}x_1$ and $C_2 = y_1y_2 \dots y_{2r+1}y_1$ and the path joining C_1 and C_2 by P . Without loss of generality, we assume that the length of P is odd (for the case of even length, the proof can be carried out similarly) and $P = z_1z_2 \dots z_{2m}$, where $z_1 = x_1$ and $z_{2m} = y_1$, and h takes alternatively 0 and 1 on edges of P with $h(z_1z_2) = 0$. By (a), we have $h(e) = \frac{1}{2}$ for every edge $e \in E(C_1 \cup C_2)$. Let $h'(x_{2i-1}x_{2i}) = h(x_{2i-1}x_{2i}) - \frac{1}{2} = 0$, $h'(x_{2i}x_{2i+1}) = h(x_{2i}x_{2i+1}) + \frac{1}{2} = 1$ ($i = 1, 2, \dots, k$); $h'(x_{2k+1}x_1) = h(x_{2k+1}x_1) - \frac{1}{2} = 0$. Similarly, let $h'(y_{2i-1}y_{2i}) = h(y_{2i-1}y_{2i}) - \frac{1}{2} = 0$, $h'(y_{2i}y_{2i+1}) = h(y_{2i}y_{2i+1}) + \frac{1}{2} = 1$ ($i = 1, 2, \dots, r$); $h'(y_{2r+1}y_1) = h(y_{2r+1}y_1) - \frac{1}{2} = 0$. Let $h'(z_{2i-1}z_{2i}) = 1$ ($i = 1, 2, \dots, m$) and $h'(z_{2i}z_{2i+1}) = 0$ ($i = 1, 2, \dots, m-1$). For the edges $e \notin C_1 \cup C_2 \cup P$, let $h'(e) = h(e)$. It is easy to verify that $|E_{h'}^*| < |E_h^*|$, a contradiction.

Conversely, suppose that $F = G_h \in \mathcal{F}_{\min}$ is a fractional (g, f) -factor of G satisfying (a) and (b). We show that $|E_h^*|$ is minimum. For $F' = G_{h'} \in \mathcal{F}_{\min}$, we have $h'(E(G)) = h(E(G))$ by Theorem 8.4.5. It is easy to see that F' can be obtained from F by exchanging the values of h on some even chains. Since there are no even decreasing chains with respect to F^* in G , we have $|E_{h'}^*| \geq |E_h^*|$. $\square$

With minor modifications in the proof of Theorem 8.4.3 and by Lemma 8.4.1, we can obtain the following theorem.

Theorem 8.4.7 (Liu and Li [356]). *Let G be a graph. Then $F = G_h$ is a minimum fractional (g, f) -factor of G with minimum number of edges if and only if $|E_h^*|$ is minimum.*

From the proof of Theorem 8.4.4, we can design a polynomial algorithm for finding a minimum fractional (g, f) -factor of a graph from fractional (g, f) -factors. Similarly, a polynomial algorithm for finding a minimum fractional (g, f) -factor with minimum number of edges from fractional (g, f) -factors is implied in the proofs of Lemma 8.4.1, Theorems 8.4.6 and 8.4.7.

More discussions about maximum and minimum fractional (g, f) -factors, in particular fractional k -factors, can be found in [370; 365].

8.5 Connected fractional factors

The study of connected factors of graphs is a relatively new and interesting topic in graph theory. It is an extension of the famous Hamiltonian cycle problem. In this section we mainly discuss connected fractional factors of graphs.

Let $F = G_h$ be a fractional (g, f) -factor of G with indicator function h . If F is connected, then we call F a *connected* fractional (g, f) -factor of G . Connected fractional f -factors, connected fractional $[a, b]$ -factors and connected fractional k -factors of a graph G can be defined in similar ways.

A *chain* L is a sequence of edges $e_1, e_2, \dots, e_m$ such that every edge appears no more than twice and the successive two edges are incident. Sometimes we write $L = v_1 v_2 \dots v_m v_{m+1}$ for convenience, where $e_i = v_i v_{i+1}$ ($1 \leq i \leq m$) and the chain L is *closed* if $v_1 = v_{m+1}$. An *increasing chain* with respect to h is a closed chain of even length with its edges having weights greater than 0 and less than 1, alternatively, and with at least one edge of weight 0. One example of increasing chains is the graph consisting of two odd cycles joint by a single edge e , where $h(e) = 0$ and the weights of all edges on two odd cycles are positive but less than one. It is easy to see that no two adjacent edges of weight zero can appear in an increasing chain. We characterize maximum fractional k -factors as follows.

Theorem 8.5.1 (Liu, Yu and Zhang [365]). *Let $F = G_h$ be a fractional k -factor of a graph G with indicator function h . Then F is a maximum fractional k -factor of G if and only if G has no increasing chains with respect to h .*

For connected fractional k -factors, we pose the following conjecture.

Conjecture 8.5.1. *Let G be a graph. If $t(G) \geq k \geq 1$, then G has a connected fractional k -factor.*

In Section 8.3, we have discussed the relationship between toughness and existence of fractional factors. Furthermore, we proved that G has a fractional 1-factor if $t(G) \geq 1$ (Theorem 8.3.2). Now we show a stronger result, which also confirm the case $k = 1$ of Conjecture 8.5.1.

Theorem 8.5.2 (Liu, Yu and Zhang [365]). *If G is a graph with $t(G) \geq 1$, then G has a connected fractional 1-factor.*

Proof. It follows from Theorem 8.3.2 that G has a fractional 1-factor. Suppose that G has no connected fractional 1-factors. Let $F = G_h$ be a maximum fractional 1-factor. By Theorem 8.5.1, there exist no increasing chains in G with respect to h . Assume that all the components of F are $F_1, F_2, \dots, F_m$ ($m \geq 2$). It is easy to see that if $F_i \not\cong K_2$, then $0 < h(e) < 1$ for each $e \in E(F_i)$ and F_i must contain a cycle.

Because $t(G) \geq 1$, we have $\kappa(G) \geq 2$. In particular, G is connected. Denote a bipartite component with bipartition (X_i, Y_i) by $F_i = (X_i, Y_i)$.

Now we construct a new graph $G' = (V', E') = (V_1 \cup V_2, E_1 \cup E_2 \cup E_3)$, where

$$\begin{aligned}
V_1 &= \bigcup_i \{x_i, y_i \mid F_i = (X_i, Y_i) \text{ is bipartite}\}, \\
V_2 &= \bigcup_i \{z_i \mid F_i \text{ is non-bipartite}\}, \\
E_1 &= \bigcup_i \{x_i y_i \mid F_i = (X_i, Y_i) \text{ is bipartite}\}, \\
E_2 &= \bigcup_i \{u_i u_j \mid E_G(U_i, U_j) \neq \emptyset, u_i \in U_i, u_j \in U_j\},
\end{aligned}$$

where $U_i = \{X_i, Y_i\}, U_j = \{X_j, Y_j\}$ ($i \neq j$). And finally

$$E_3 = \bigcup_i \{u_i z_j \mid E_G(U_i, F_j) \neq \emptyset, F_j \text{ is non-bipartite}, u_i \in U_i, z_j \in F_j\}.$$

The edges in E_1 are called *F-edges* and the edges in $E_2 \cup E_3$ are called *G-edges*. Clearly, each *F*-edge corresponds to an odd path in a bipartite component of G_h and each *G*-edge corresponds to the edge(s) in $E(G) - E(F)$. An (F, G) -alternating path in G' is a path whose edges are alternately *F*-edges and *G*-edges. It is not hard to verify that a closed alternating path of even length in G' corresponds to an increasing chain with respect to h in G .

Let $P_{G'} = u_1 u_2 \dots u_k$ be a longest (F, G) -alternating path in G' . Then $P_{G'}$ corresponds to an alternating path P_G with respect to h in G . Moreover, if the end-vertex u_1 of $P_{G'}$ is incident with a *G*-edge, then $u_1 \in V_2$. Next we consider three cases.

Case 1. $u_1 \in V_2$ and $u_k \in V_2$.

Then u_1 is connected to an odd cycle C_1 of F_1 by a path P_{u_1} , and u_k is connected to an odd cycle C_k of F_k by a path P_{u_k} . Then $C_1, P_{u_1}, P_G, P_{u_k}, C_k$ form an increasing chain in G with respect to h (see Fig. 8.7), a contradiction.

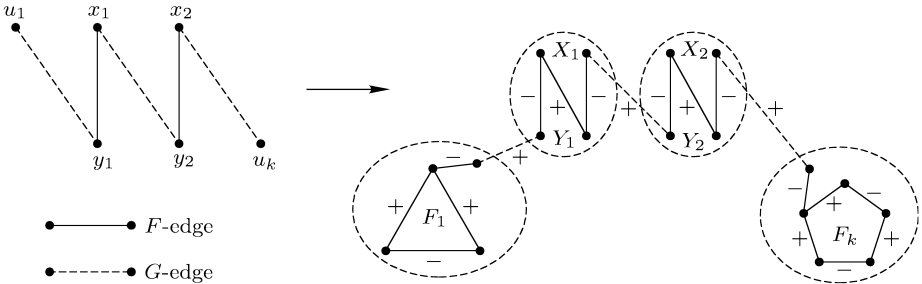


Fig. 8.7 Illustration of Case 1, where '+' and '-' represent weights greater than zero and less than one, respectively.

Case 2. $u_1 \in V_1$ and $u_k \in V_1$.

Without loss of generality, assume $u_1 = x_1$ and $u_k = y_k$. Then either X_1 or Y_k is independent in G . Otherwise, let e_1 be an edge joining two vertices in X_1 and e_k be an edge joining two vertices in Y_k . Denote the odd cycles containing e_1 and e_k by C_1 and C_k , respectively. Then C_1, P_G, C_k form an increasing chain in G with respect to h , a contraction. Assume that X_1 is independent in G . If a vertex v of X_1 is adjacent to a vertex w outside of the path P_G , then the edge vw corresponds to a G -edge of G' which contradicts to the fact that $P_{G'}$ is a longest (F, G) -alternating path. If a vertex v of X_1 is adjacent to a vertex w on the path P_G , then there is a cycle C_1 in G containing the edge vw . If the cycle C_1 is even, then it is an increasing chain in G , a contradiction. If C_1 is odd, then Y_k must be an independent set in G . Use the similar arguments as to X_1 , we see that no vertex in Y_k is adjacent to a vertex outside P_G and the cycle C_k containing an edge from Y_k to a vertex of P_G is odd. Thus C_1, C_k and an alternating path form an increasing chain in G again, a contradiction. Therefore, the vertices in Y_k is not adjacent to other vertices besides those in X_k . In other words, X_k is a cut-set of G . By Corollary 8.2.4, we have $|X_k| = |Y_k|$. Let $S = X_k$. Then $\omega(G - S) \geq |Y_k| + 1 = |S| + 1$. But

$$1 \leq t(G) \leq \frac{|S|}{\omega(G - S)} \leq \frac{|S|}{|S| + 1} < 1,$$

a contradiction.

Case 3. $u_1 \in V_1$ and $u_k \in V_2$.

Without loss of generality, assume that $u_1 = x_1$. The proof in this case is similar to that of Case 2. First, we show that X_1 is independent in G . Otherwise, let e_1 be an edge joining two vertices in X_1 and C_k be an odd cycle in F_k connected by a path P_k to the path P_G . Then the odd cycle C_1 , the paths P_G, P_k and the cycle C_k form an increasing chain in G with respect to h , a contraction. If a vertex v of X_1 is adjacent to a vertex w outside of the vertices of the path P_G , then the edge vw corresponds to a G -edge of G' which contradicts to the fact that $P_{G'}$ is a longest (F, G) -alternating path. Next, if a vertex v of X_1 is adjacent to a vertex w on the path P_G , then it creates a cycle C_1 in G with odd length. Otherwise, if the cycle C_1 has even length, then it is an even alternating cycle and thus an increasing chain in G , a contradiction. Then C_1 and an odd cycle in the non-bipartite component F_k together with an alternating path form an increasing chain in G again, a contradiction. Therefore, the vertices in X_1 is not adjacent to other vertices besides those in Y_1 , or in other words, Y_1 is a cut-set of G . By Corollary 8.2.4, we have $|X_1| = |Y_1|$. Let $S = Y_1$. Then

$$1 \leq t(G) \leq \frac{|S|}{\omega(G - S)} \leq \frac{|S|}{|S| + 1} < 1,$$

a contradiction.

The proof is completed. $\square$

We have shown that the condition $t(G) \geq 1$ is tight in Theorem 8.3.2, so it is also tight in Theorem 8.5.2. Next we present properties of connected fractional 1-factors.

Theorem 8.5.3 (Liu, Yu and Zhang [365]). *Let G_h be a connected fractional 1-factor of a graph G . If G_h is bipartite, then G_h is 2-connected.*

Proof. Let $G_h = (X, Y)$ be a bipartite connected fractional 1-factor of G . Then it follows from Corollary 8.2.4 that $|X| = |Y|$. Suppose that $v \in X$ is a cutvertex of G_h and $G_i = (X_i, Y_i)$, $1 \leq i \leq m$, are all the components of $G_h - \{v\}$. Obviously, $m \geq 2$ and $N(v) \cap Y_i \neq \emptyset$ for each i , $i \in \{1, 2, \dots, m\}$, where $N(v)$ denotes the set of neighbors of v in G_h . However, for any G_i , we have

$$\sum_{e \in E(G_i)} h(e) = \sum_{v \in X_i} h(E_v) = \sum_{v \in X_i} 1 = |X_i|.$$

Similarly,

$$\sum_{e \in E(G_i)} h(e) < \sum_{v \in Y_i} h(E_v) = \sum_{v \in Y_i} 1 = |Y_i|.$$

Thus $|X_i| \leq |Y_i| - 1$. But

$$|X| = \sum_{i=1}^m |X_i| + 1 \leq \sum_{i=1}^m (|Y_i| - 1) + 1 = |Y| - m + 1 < |Y|,$$

a contradiction. Therefore, G_h has no cutvertices or G_h is 2-connected. $\square$

Theorem 8.5.4 (Liu, Yu and Zhang [365]). *Let G be a graph of order at least 5. If G has a connected fractional 1-factor G_h with minimum number of edges, then G_h has no 4-cycles.*

Proof. Let G be a graph of order at least 5 and let G_h be a connected fractional 1-factor with minimum number of edges in G . We prove that G_h has no 4-cycles. Suppose that G_h has a 4-cycle $C = v_1 v_2 v_3 v_4 v_1$.

We consider two cases.

Case 1. The cycle C contains at least one chord.

Suppose that $v_1 v_3 \in E(G_h)$. Since $f(v_1 v_3) > 0$, then at least one of v_2 and v_4 is of degree at least 3 in G_h . If only one of them, say v_4 , is of degree at least 3 in G_h , then $\varepsilon = f(v_3 v_4) < f(v_1 v_2)$. Starting from $v_1 v_2$, we alternately decrease and increase the weights on C by ε and then it results in a connected fractional 1-factor of G with less edges (we say that G_h can be *improved through* C from now on), a contradiction to the choice of G_h . Similarly, if both v_2 and v_4 are of degree at least 3 in G_h , then G_h can be improved through C , a contradiction again.

Case 2. The cycle C contains no chords.

There is at least one pair of adjacent vertices on C , say v_1 and v_2 , that have neighbors outside C . If v_4 or v_3 is of degree 2 in G_h , say $d_{G_h}(v_4) = 2$, then $f(v_1v_2) < f(v_3v_4)$. Let $f(v_1v_2) = \varepsilon$. Replacing $f(v_2v_3)$ and $f(v_1v_4)$ by $f(v_2v_3) + \varepsilon$ and $f(v_1v_4) + \varepsilon$, and $f(v_1v_2)$ and $f(v_3v_4)$ by 0 and $f(v_1v_4) - \varepsilon$. Then we obtain a connected fractional 1-factor with one less edge, a contradiction. Suppose that $d_{G_h}(v_3) \geq 3$ and $d_{G_h}(v_4) \geq 3$. If $f(v_1v_2) \neq f(v_3v_4)$, $f(v_1v_4) \neq f(v_2v_3)$ or $G_h - \{v_1v_2, v_3v_4\}$ or $G_h - \{v_1v_4, v_2v_3\}$ is connected, then G_h can be improved through C . Otherwise, $f(v_1v_2) = f(v_3v_4)$, and $f(v_1v_4) = f(v_2v_3)$ and both $G_h - \{v_1v_2, v_3v_4\}$ and $G_h - \{v_1v_4, v_2v_3\}$ are disconnected. Let G_1 and G_2 be the components of $G_h - \{v_1v_4, v_2v_3\}$ such that v_1v_2 and v_3v_4 are in G_1 and G_2 , respectively. If one of the components, say G_1 , is bipartite, then v_1v_2 is contained in an even cycle C_1 of G_1 . We can alternately increase and decrease a small positive amount on the edges of C_1 such that $f(v_1v_2)$ decreases. Then G_h can be improved, a contradiction. So G_1 and G_2 are non-bipartite. Then we can form a closed even chain W by using two odd cycles, from G_1 and G_2 respectively, and a path through v_1v_4 connecting them. Now we can adjust on W to make $f(v_1v_4)$ decrease and then G_h can be improved again. This last contradiction leads to the conclusion of the proof. $\square$

At the end of this chapter, we leave a few open problems to stimulate the further research on this topic.

Open Problem 8.5.1. *Find more properties of connected fractional k -factors with minimum number of edges.*

Open Problem 8.5.2. *When is a connected fractional 1-factor of a bipartite graph with minimum number of edges a minimal 2-connected graph?*

Open Problem 8.5.3. *Find characteristics and properties of connected fractional (g, f) -factors of a graph.*

Open Problem 8.5.4. *Investigate sufficient conditions for existence of connected fractional (g, f) -factors.*

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